CHAPITRE II
DEFORMATIONS

CHAPITRE II : LES DEFORMATIONS

I. Introduction

II. Gradient d'une transformation

! " #

$% $ % "
$ " $ # " $
$ " $ # $

III. Tenseur des dilatations

& ' $ () '
' $ () ' * $

IV. Déformations

+ '),
+ #)-
+ . '), #)-

+ $
+ - $
+ ' $

V. Formulation en fonction des déplacements

+
+ $ $ ((
+ = = $ =

+ / $ $
+ $ - $
+ $ $
+ 0 $ 1
+
+ 2 " 3

VI. Déformation en petite transformation

+
+ $

+ 1 $ $ $
+ $ # $ $ 4$ $ 3

VII. Vitesses de déformation

=

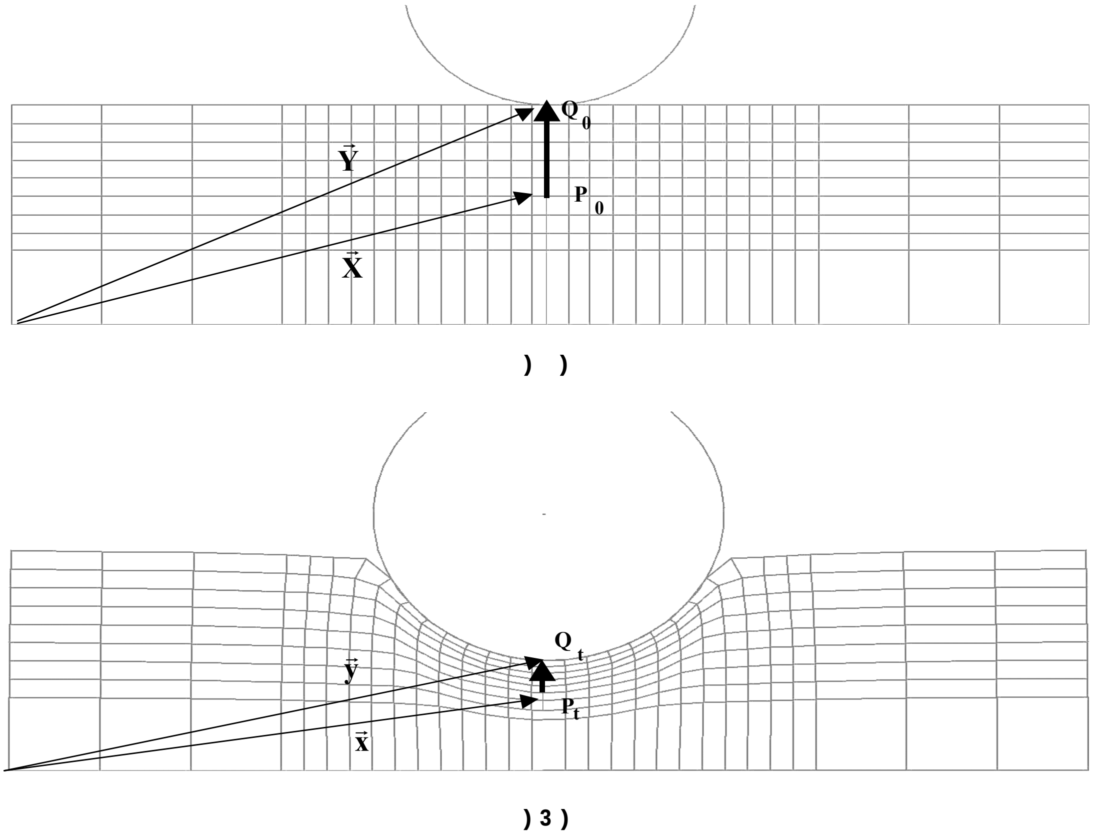


Figure 1 : Réseau de lignes dessinées sur un solide (a) avant indentation et (b) après indentation par un cylindre.

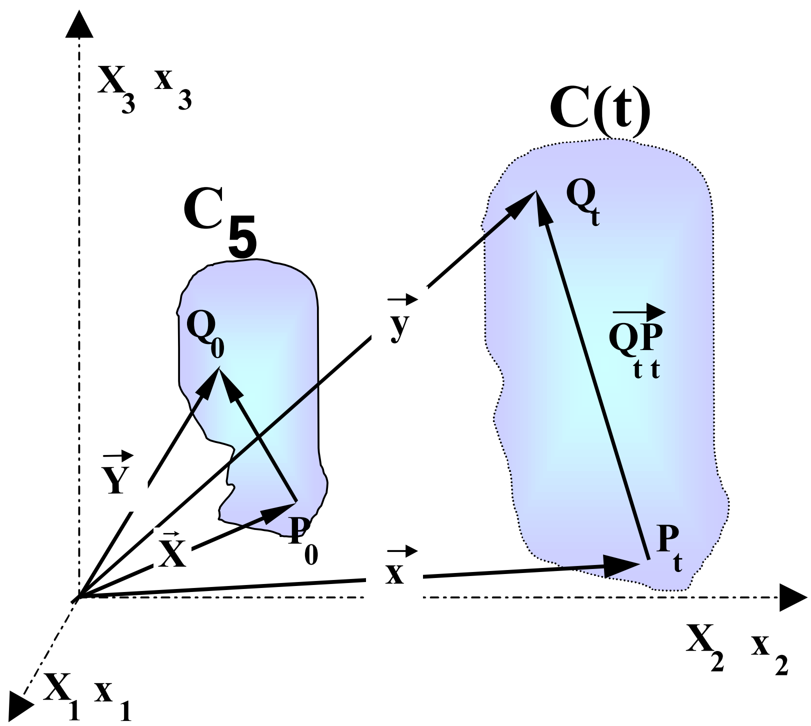


Figure 2 : Configuration initiale et actuelle d'un solide déformé.

I. INTRODUCTION

, # # $(" $

$ $ $ 6 $

- 7 $ 07 88%

- # $ 9
- " 9
- "

$ $ %

" 0 " : $ %

" "

II. GRADIENT D'UNE TRANFORMATION

II.1. Tenseur gradient d'une transformation

Aα $ " 5 > " = B%% 7*i.e* &5 " <5 8
6 " $ $ 4

$$\vec{y}-\vec{x}=\underline{\underline{F}}\left(\vec{X}\%t\right)\left(\vec{Y}-\vec{X}\right)+\vec{\alpha}\left(\vec{Y}-\vec{X}\%t\right)\left\|\vec{Y}-\vec{X}\right\|$$
$$\overrightarrow{P_tQ_t}=\underline{\underline{F}}\left(5P_5\%t\right)\overrightarrow{P_5Q_5}+\vec{\alpha}\left(\overrightarrow{P_5Q_5}\%t\right)\left\|\overrightarrow{P_5Q_5}\right\| \qquad 7\ 8$$

, $\underline{\underline{E}}7\vec{=}\%8=\overrightarrow{}D\vec{\Phi}7\vec{=}\%8C$ $\underline{\underline{E}}$
$ <5 , $ $\underline{\underline{E}}$ 4

$$\underline{\underline{F}}7\vec{X}\%t8=\overrightarrow{grad}D\vec{\Phi}7\vec{X}\%t8C=\overrightarrow{grad}D\vec{x}7\vec{X}\%t8C$$
$$F_{ij}=\frac{\partial x_i}{\partial X_j} \qquad 7\ 8$$

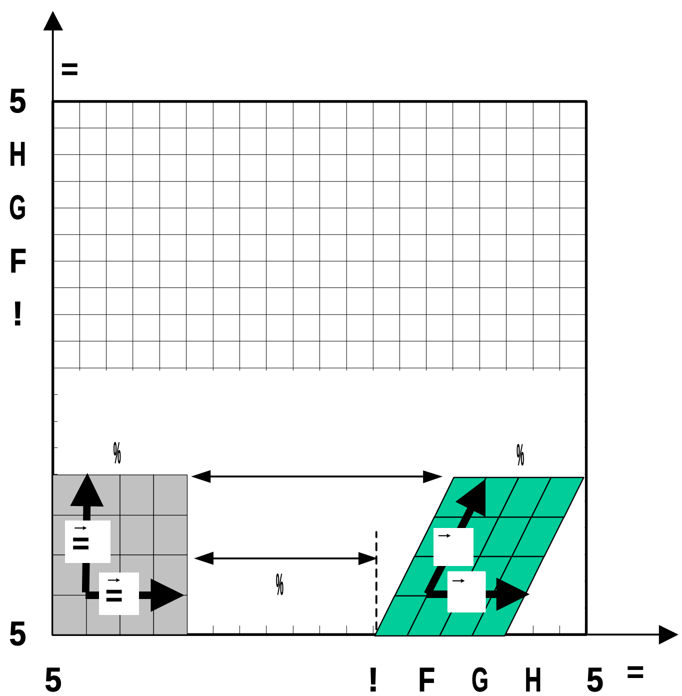


Figure 3 : Exemple de transformation particulière : glissement simple avec translation

6 $\underline{\underline{E}}$ &5%

transformation homogène :

$$\vec{x} = \vec{\Phi}(\vec{X},t) = \underline{\underline{F}}(t)\vec{X} + \vec{B}(t)$$

$$x_i = \Phi_i(X_1, X_2, X_3, t) = F_{ij}(t)X_j + B_i(t)$$

7 8

, 7 8 #$ $ # $\overrightarrow{\&<} = \underline{\underline{E}}(\,)\,\overrightarrow{\&5<5}$

, $ $\vec{\alpha}$

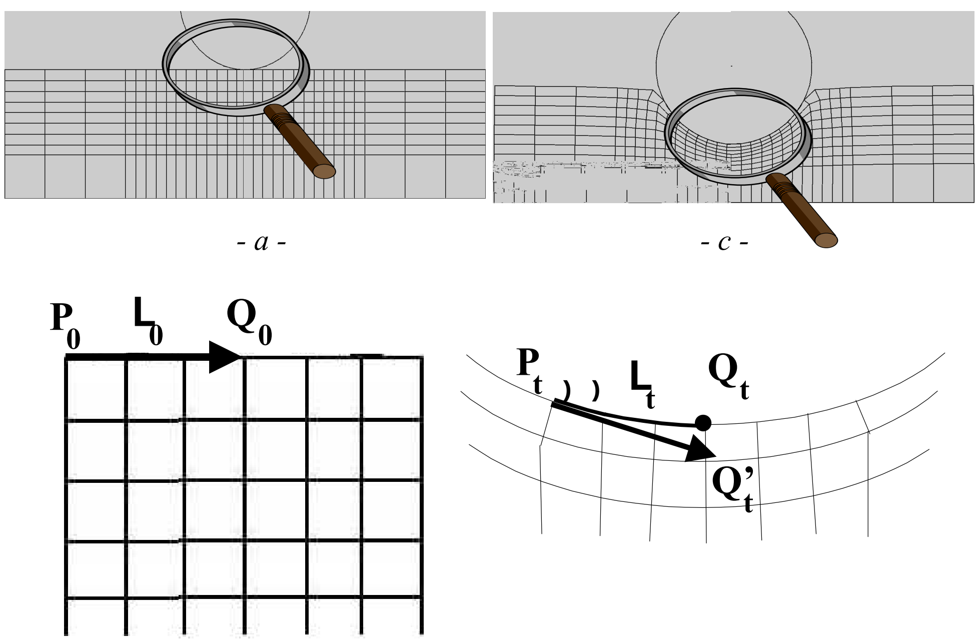


Figure 4: Indentation d'un solide par un cylindre infinie en configuration non déformée prise comme référence (a et b) et configuration déformée (c et d).

, # # $(3 ,

3 $ * $ 0 $

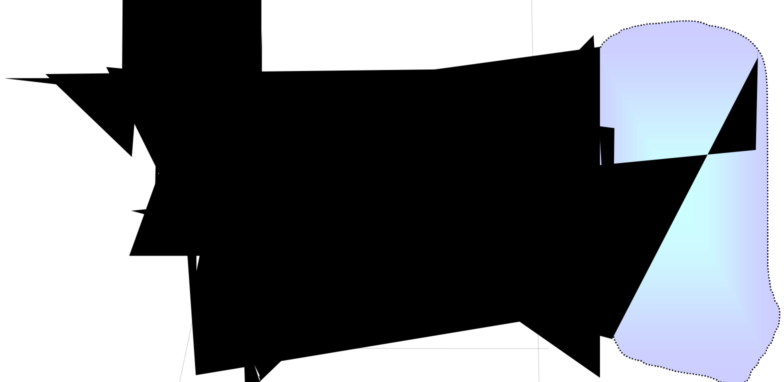
$ $ C5 , $ $ * $

[redacted]

...ansporté du vecteur $\overrightarrow{P_5Q_5}$

$\overrightarrow{P_tQ}_t = \underline{\underline{F}}\left(P_5, t\right)\,\overrightarrow{P_5Q_5}$

$\overrightarrow{P_tQ_t} = \overrightarrow{P_tQ}_t + \vec{\alpha}\left(\overrightarrow{P_5Q_5}, t\right)\left\|\overrightarrow{P_5Q_5}\right\|$



II.2. Transport convectif

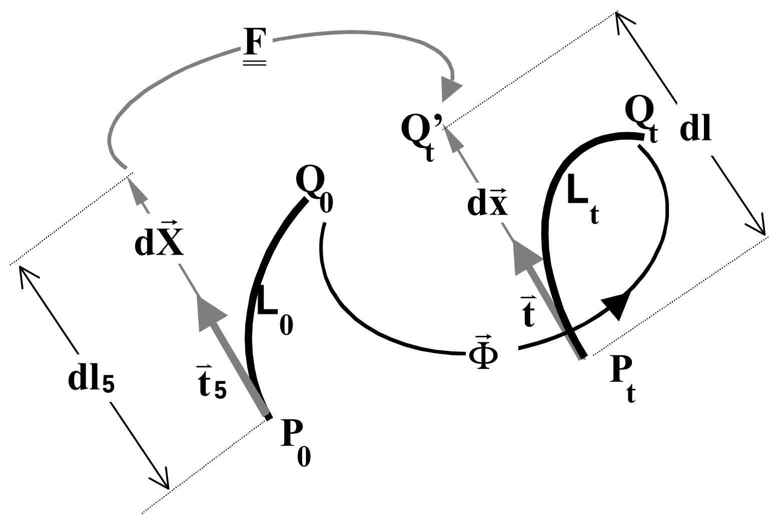


Figure 6 : transport d'un vecteur élémentaire

" $ $ → - = → ∧ →J #$ $ %

- = ε ?M ? JM % A (3 # (7 - 8 $ 4

ε ?M = 5 $ 9

ε ?M = % ?% M $ 7 % % 8 9

ε ?M = − % ?% M $ 7 % % 8

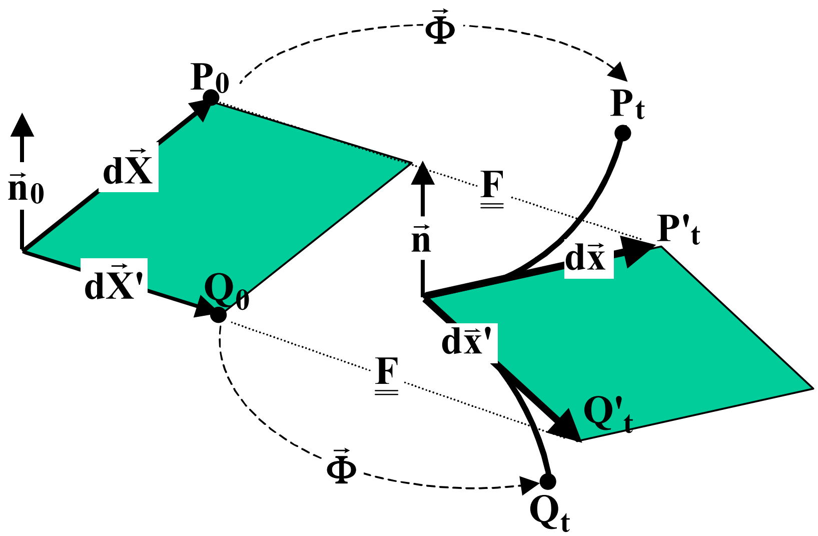


Figure 7: Transport d'un élément de surface

I → E = 7 J? = E? =J 8 →J E =J 7 JM = EM =J 8

$ % 3

- = ε ?M (E? =)(EM =J)

& K # E% $N

E # $

E - = ε ?M E E? EM = =J

; ε ?M E E? EM = ε 7E8 = ε 0 7 - 8% $

E - = ε 0 = =J

0 # - $ - 5 →5 % = ∧ =J = →5 - 5

#$ $ 5 - 5 = ε = =J 0 #$ 4

E - = 0 5 - 5

E = E = [E →]

, $ $ #$

E → - = 0 → 5 - 5 %

→ - = 0 ' → 5 - 5 ' = E — 7G8

$ $ 0 5 % $ $ 3 , 5 &5 -
" &5 $ $ $ " $ = = 6
= = &5 +5 = % = 07 8% , # $, 5 $ " $
" $ → % " $ = , 6 $ 65 $
" $ $ " $ → → % = =

III. TENSEUR DES DILATATIONS

0 " 3 # # $(
7E H8 $
$ $ $ $, "
# $ $ $,
* 3 3 $: $ " $
% $ $ # " " $,
$ # " $) : # , $
" $ # # $ " $

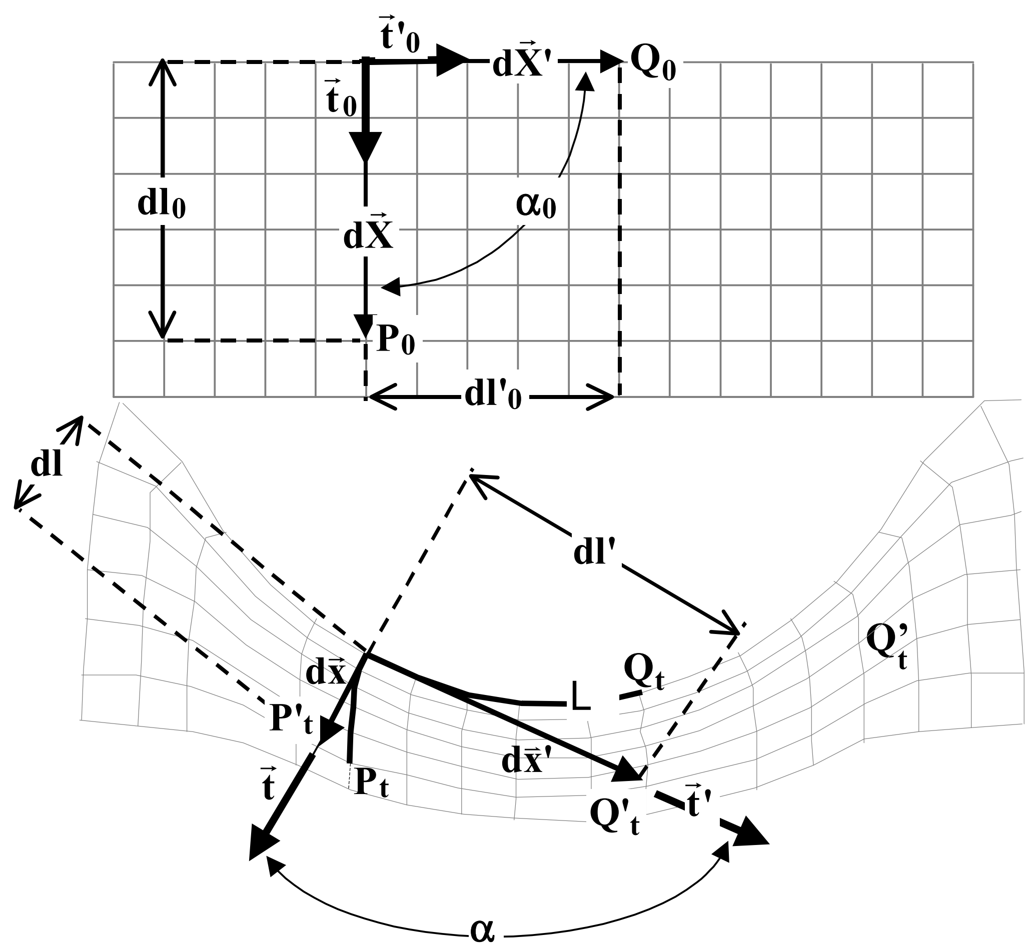


Figure 9: Indentation d'un solide par un cylindre infinie. Configuration de référence et configuration déformée.

III.1. Tenseur des dilatations

0 $ $

$$\underline{\underline{E}}\;\underline{\underline{E}}$$

$$\underline{\underline{0}} = \underline{\underline{E}}\;\underline{\underline{E}}$$

7H8

, $\underline{\underline{0}} = \underline{\underline{E}}\;\underline{\underline{E}}$ ' $ () ' 0

$ $ $ $ " $

$ $ $ " $ $

$ 0 % % # " $

* $ % $\underline{\underline{0}}$, $

$\underline{\underline{0}}$

$\underline{\underline{0}}$ (7 8 4

$$\underline{\underline{0}} = \left(\underline{\underline{E}}\;\underline{\underline{E}}\right) = \underline{\underline{E}}\left(\underline{\underline{E}}\right) = \underline{\underline{E}}\;\underline{\underline{E}} = \underline{\underline{0}}$$

, 3 $\underline{\underline{0}}$ " %

$$\left[\underline{\underline{0}}\right] = \left[\underline{\underline{E}}\;\underline{\underline{E}}\right] = \left(\underline{\underline{E}}\right) = 0 > 5 \quad \underline{\underline{0}} = \| \; \| > 5$$

' $ (– ' * $ 7 58

$\underline{\underline{I}} = \underline{\underline{E}}\underline{\underline{E}}$

, # " 0 $ ( ) ' * $ % $ $

3 " $ $ $ 4

$$\underline{\underline{\$}} = \left(\underline{\underline{I}}\right)^{-} = \left[\underline{\underline{E}}\underline{\underline{E}}\right]^{-} = \left[\underline{\underline{E}}^{-}\right] \underline{\underline{E}}^{-} = \underline{\underline{'}} \; \underline{\underline{'}} \qquad \$$$

$$\vec{\underline{\underline{E}}} \quad \vec{\underline{\underline{E}}}' = \; \vec{} \; \underline{\underline{\$}} \; \vec{}'$$

$$\underline{\underline{\$}} = \underline{\underline{I}}^{-} = \underline{\underline{'}} \; \underline{\underline{'}}$$

$$\lambda(\vec{=}) = \lambda(\vec{\ }_5) \begin{cases} = \sqrt{\vec{\ }_5 \, \underline{\underline{0}} \, \vec{\ }_5} \\ = \frac{}{5} \\ = \left\| \underline{\underline{E}} \, \vec{\ }_5 \right\| \end{cases}$$ $ (7 8)

6 # $ K 0 &5% $ $ $ λ
$ = &5 λ $ = %
&5 # 0 # $ % $ 0 ;
$ 5 : λ7 5 8 * $ $
$ $ $ 6 5 $ *
" $ 3 % 4 $\lambda(\vec{=}) = \sqrt{\vec{\ } \, \underline{\underline{0}} \, \vec{\ }} = \sqrt{0\_}$, # $ %
#(

III.3. Glissement de deux directions orthogonales

6 H% $ $,
" $ = = α5 $ $
$ $ % " $ α

$$\$ \quad \alpha = \frac{\vec{\ } \ \vec{\ }}{\| \vec{\ } \| \| \vec{\ } \|} = \frac{(\underline{\underline{E}} \, \vec{=}) \, \underline{\underline{E}} \, \vec{=}}{\sqrt{(\vec{=} \, \underline{\underline{0}} \, \vec{=})} \sqrt{(\vec{=} \, \underline{\underline{0}} \, \vec{=})}}$$

$$\$ \quad \alpha = \frac{\vec{\ } \ \vec{\ }}{\| \vec{\ } \| \| \vec{\ } \|} = \frac{\vec{=} \, (\underline{\underline{E}} \ \underline{\underline{E}}) \, \vec{=}}{\sqrt{(\vec{=} \, \underline{\underline{0}} \, \vec{=})} \sqrt{(\vec{=} \, \underline{\underline{0}} \, \vec{=})}}$$

$$\$ \quad \alpha = \frac{\vec{\ } \ \vec{\ }}{\| \vec{\ } \| \| \vec{\ } \|} = \frac{(5 \vec{\ } 5) \, (\underline{\underline{0}}) (5 \vec{\ } 5)}{\sqrt{[(5 \vec{\ } 5) \, \underline{\underline{0}} (5 \vec{\ } 5)]} \sqrt{[(5 \vec{\ } 5) \, \underline{\underline{0}} (5 \vec{\ } 5)]}}$$

$$\$ \quad \alpha = \frac{\vec{\ } \quad \vec{\ }}{\| \vec{\ } \| \| \vec{\ } \|} = \frac{\vec{5}\,\underline{\underline{0}}\,\vec{5}}{\sqrt{\vec{5}\,\underline{\underline{0}}\,\vec{5}}\sqrt{\vec{5}\,\underline{\underline{0}}\,\vec{5}}}$$

$$\$ \quad \alpha = \frac{\vec{\ } \quad \vec{\ }}{\| \vec{\ } \| \| \vec{\ } \|} = \frac{\vec{5}\,\underline{\underline{0}}\,\vec{5}}{\lambda 7 \vec{5} 8 \lambda 7 \vec{5} 8} \qquad 7 \quad 8$$

IV. DEFORMATIONS

IV.1. Tenseurs des déformations

$ $% #)- #

3 # " $ * $ $, $

3 . 4

7 !8

& 3 '),

)- % $

$ $ $ $ $ $ $

4

3 $ $ 0 $ $

* 4

$

IV.2. Déformation dans une direction , allongement unitaire dans une direction, glissement de deux directions orthogonales

Définition

? 3 $ # * # $

$ $ % " " " 3

" $ $

Définition de l'allongement unitaire dans la direction

$$\delta(\,)=\frac{\| \| - \| \|}{\| \|}=\frac{-}{}$$

, # $, #

: * $\lambda()$ $ 4

$$\delta()=\frac{-}{}=\frac{}{}-$$

$

$$\delta()=\lambda()-$$
$$=\sqrt{+\ }-$$
(7 H8)

;

" $ * # , 7 8

" $ $ $ $ 4

$$\alpha=\frac{}{\| \|\,\| \|}=\frac{}{\lambda()\lambda()}$$

, # $ 0 $ () '

$\lambda()=\sqrt{}$ " $ $\lambda()=\sqrt{}$ " $ - $ (α)

$ 4

$$ \$ \quad \alpha = \frac{5\,0\,5}{\sqrt{5\,0\,5}\sqrt{5\,0\,5}} $$

, # 7 8 '), $

0 $ ()' 0 #$ 4

$$ \$ \quad \alpha = \frac{5(\;+\;)5}{\sqrt{5(\;+\;)5}\sqrt{5(\;+\;)5}} = \frac{5\,5+5\,5}{\sqrt{+\;5\,5}\sqrt{+\;5\,5}} \quad ; \quad 5\,5 = 5\% \quad \$\,4 $$

$$ \$ \quad \alpha = \frac{5\,5}{\sqrt{+\;5\,5}\sqrt{+\;5\,5}} \qquad 7\;58 $$

$ $ A " $ 5 5 $ * " $ 3

$ $ % $5 = \;\; 5 = \;?7 \neq ?8$ 4

$$ \$ \quad \alpha = \frac{\;?}{\sqrt{+\;\;}\sqrt{+\;?\;?}} = \frac{?}{\sqrt{+}\sqrt{+\;??}} $$

A # $ *

V. FORMULATION EN FONCTION DES DEPLACEMENTS

V.1. Tenseur gradient du déplacement

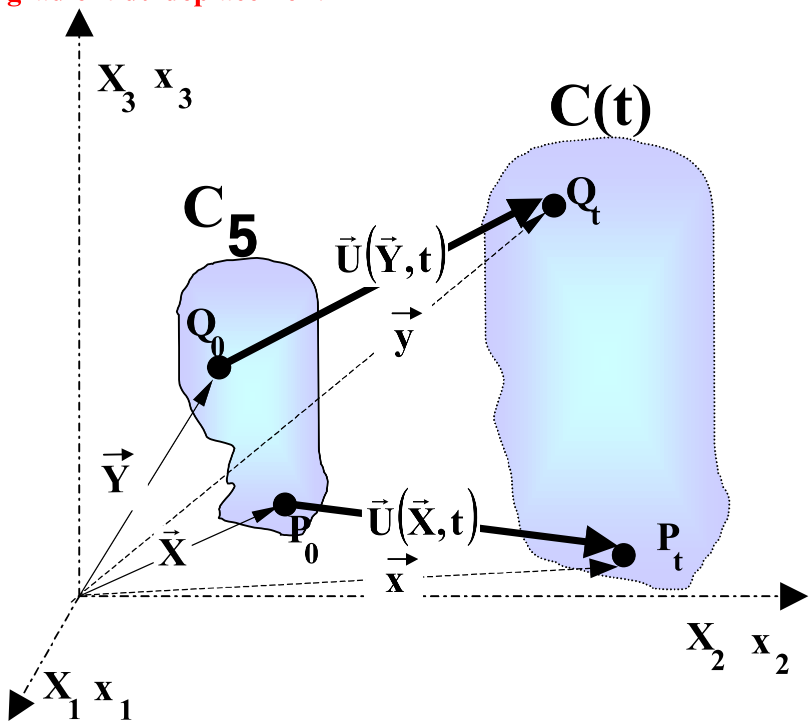


Figure 10: Configuration initiale C_0 et actulle $C(t)$ d'un solide avec représentation des vecteurs déplacements.

; $ & * # % " $

1(=%) = &5& = ;& − ;&5 = 7=% 8 − =

, $ $ $

$$\underline{\underline{H}} = \overrightarrow{grad}\left[\vec{U}\left(\vec{X},t\right)\right] \quad soit \quad H_{ij} = \frac{\partial U_i}{\partial X_j}$$ 7 8

, $ Q % $ 1 = #

0 1(=%) = 7=% 8 − =% $ Q

E 4

Q = E − 7 8

$$\% \, \underline{\underline{Q}} = \longrightarrow [\vec{1}(\vec{=}\%)] = \longrightarrow [\vec{\ } 7 \vec{=} \% \, 8 - \vec{=}] = \longrightarrow [\vec{\ } 7 \vec{=} \% \, 8] - \underline{\underline{\ }} = \underline{\underline{E}} - \underline{\underline{\ }}$$

$$\underline{\underline{\ }} = \underline{\underline{\varepsilon}} + \underline{\underline{\varpi}}$$

- $ #$ I #

(- # (- 4 - = - + - ,

(- (- 4

$$\underline{\underline{-}} = -(\underline{\underline{-}} + \underline{\underline{-}}) \quad \underline{\underline{-}} = -(\underline{\underline{-}} - \underline{\underline{-}})$$

, # $ $ $ $ $\underline{\underline{Q}}\%$

$\underline{\underline{\varepsilon}}$ $\underline{\underline{\omega}}$ 4

$$\underline{\underline{\varepsilon}} = -(\underline{\underline{Q}} + \underline{\underline{Q}}) \qquad \underline{\underline{\omega}} = -(\underline{\underline{Q}} - \underline{\underline{Q}}) \qquad 7 \quad 8$$

$ 3 .% " 4

$$\varepsilon_{?} = -\left(\frac{\partial 1}{\partial =_{?}} + \frac{\partial 1_{?}}{\partial =}\right) \qquad \omega_{?} = -\left(\frac{\partial 1}{\partial =_{?}} - \frac{\partial 1_{?}}{\partial =}\right) \qquad 7 \quad 8$$

Expression du tenseur des dilatations de Cauchy-Green $\underline{\underline{\ }}$ *en fonction du tenseur gradient des déplacements* $\underline{\underline{\ }}$

0 $\underline{\underline{E}} = \underline{\underline{\ }} + \underline{\underline{Q}}$

$$\underline{\underline{0}} = \underline{\underline{E}} \; \underline{\underline{E}}$$

$$\underline{\underline{0}} = \underline{\underline{E}} \; \underline{\underline{E}} = (\underline{\underline{\ }} + \underline{\underline{Q}})(\underline{\underline{\ }} + \underline{\underline{Q}}) = \underline{\underline{\ }} + \underline{\underline{Q}} + \underline{\underline{Q}} + \underline{\underline{Q}} \; \underline{\underline{Q}}$$

$$\underline{\underline{0}} = \underline{\underline{\ }} + \underline{\underline{\varepsilon}} + \underline{\underline{Q}} \; \underline{\underline{Q}}$$

Expression du tenseur des déformations de Green-Lagrange $\underline{\underline{\ }}$ *en fonction du tenseur gradient des déplacements* $\underline{\underline{\ }}$

0 $\underline{\underline{\ }} = (\underline{\underline{0}} - \underline{\underline{\ }})$

$$\underline{\underline{\ }} = -(\underline{\underline{0}} - \underline{\underline{\ }}) = \underline{\underline{\varepsilon}} + -\underline{\underline{Q}}\ \underline{\underline{Q}} \qquad 7 \quad 8$$

V.2. Décomposition du déplacement

6 &5 $, 0 $ ()
' $\underline{\underline{0}}$ ($ 3 , 3 " $ $\underline{\underline{0}}$
3 $ % ')
, $\underline{\underline{\ }}$ * : " $ $\underline{\underline{0}}$ 6 $\vec{l}$ % $\vec{l}$
$\vec{l}$ " $ $\underline{\underline{0}}$ R %R %R " $ % 4

$$\underline{\underline{0}}\,\vec{l} = \mu\,\vec{l}$$

$$\underline{\underline{\ }}\,\vec{l} = -(\underline{\underline{0}} - \underline{\underline{\ }})\,\vec{l} = \left(\frac{R}{\ } - \right)\vec{l}$$

0 $\underline{\underline{0}}$ (% " $\underline{\underline{0}}$ "

$$\underline{\underline{0}}\,\vec{l} = \mu\,\vec{l} \qquad (= \% \%)$$

$ $\vec{l}$ 4

$$\vec{l}\ \underline{\underline{0}}\ \vec{l} = R\ \vec{l}\ \vec{l} = R \qquad (= \% \%)$$

, 3 $ $

$$\lambda(\vec{l}) = \sqrt{\vec{l}\ \underline{\underline{0}}\ \vec{l}} = R\ \vec{l}\ \vec{l} = \sqrt{R} \qquad (= \% \%)$$

$$\vec{3} = \underline{\underline{E}}\ \vec{l} \qquad (= \% \%)$$

$$\vec{3}\ \vec{3}_? = \vec{l}\ \underline{\underline{E}}\ \underline{\underline{E}}\ \vec{l}_?$$

$$\vec{3}\ \vec{3}_? = \vec{l}\ R_?\ \vec{l}_?$$

$\vec{3}\ \vec{3}_? = R_? \delta_?$

$$\vec{3}\ \vec{3}_? = 5 \quad \neq ? \qquad \|\vec{3}\| = R \quad \|\vec{3}\| = \lambda \qquad \text{7 !8}$$

6

$$\underline{\underline{0}} = \begin{bmatrix} R = \lambda & 5 & 5 \\ 5 & R = \lambda & 5 \\ 5 & 5 & R = \lambda \end{bmatrix}$$

, '), = : $
$ 0 $ ()' $\underline{\underline{0}}$;
$ % " = , $ *
$ $ 4

$$= -(R -) = -(\lambda -) \geq - - (= \% \%) \qquad \text{7 F8}$$

, " R " %
$ $ * S @ , $ = #$ 3
$ 4

$$= = \begin{bmatrix} = -(\lambda -) & 5 & 5 \\ 5 & = -(\lambda -) & 5 \\ 5 & 5 & = -(\lambda -) \end{bmatrix}$$

, # $ % # " $ $\vec{l}$
$ $ * $ $ 4

$$\delta = \delta(\vec{l}) = \lambda - = \sqrt{R - } = \sqrt{ + } - \quad \text{7 G8}$$

0 &5 $ $ &

$ $) &5 *

& $ $ $ % $ $ $

% " $ # 3 & 3 $

" 4

$$\vec{3} = \underline{\underline{E}}\,\vec{l} \quad \vec{l}$$

0 % " $ * % $

l , . $ $

$ % 3 $\{\vec{l}\ \%\vec{l}\ \%\vec{l}\ \}$ 3 $\{\vec{3}\ \%\vec{3}\ \%\vec{3}\ \}\%$

: 0# $ % # A

. , 1 3 $ $

" $ $\vec{l}$ " $ $ *) : * $

$\vec{3}$, $ 1 $ $ $;

1 $ * , +

% $ $ $ 0

$ " $ $\vec{3} = \underline{\underline{E}}\,\vec{l}\ \ l$ #

: $ % 1 + : " $\lambda = \|\vec{3}\| = \sqrt{R}$ 0

$ $ $ $ * $

$ % $ 4

$$\underline{\underline{\ \ }} = \underline{\underline{\ \ }}\,\underline{\underline{\ \ }} = \underline{\underline{\ \ }}\,\underline{\underline{\ \ }} \quad \text{7 H8}$$

0 ? $ " T * U

T * $ U 1 + ; T * U

T * $ U, $ $\underline{\underline{1}}$ 3 $ $\{\bar{I}\ \%\bar{I}\ \%\bar{I}\ \}$

4

$$\underline{\underline{1}} = \begin{bmatrix} \lambda & 5 & 5 \\ 5 & \lambda & 5 \\ 5 & 5 & \lambda \end{bmatrix}$$

0# : $ $ $\underline{\underline{+}}$ 3 $\{\bar{3}\ \%\bar{3}\ \%\bar{3}\ \}$ 1

" $ 3 $ " $ $ 0 %

" $ $ $ " $ " $

$ 1 " $ $\vec{\equiv}$ $ $ 3

*

, 0 $ ()' #$ $\underline{\underline{0}} = \underline{\underline{E}}\ \underline{\underline{E}}$ $

$\underline{\underline{.}}$ $\underline{\underline{1}}\ \%\ \underline{\underline{0}}$ #$ 4

$$\underline{\underline{0}} = \left(\underline{\underline{.}}\,\underline{\underline{1}}\right)\ \underline{\underline{.}}\,\underline{\underline{1}} = \underline{\underline{1}}\ \underline{\underline{.}}\ \ \underline{\underline{.}}\,\underline{\underline{1}}$$

; $\underline{\underline{.}}$,# " $\underline{\underline{.}}$ $\underline{\underline{.}}$ $\underline{\underline{1}}$

($\underline{\underline{.}}\ \underline{\underline{.}} = \underline{\underline{1}}\ \underline{\underline{1}} = \underline{\underline{1}}\ \underline{\underline{1}}$, 0 $ ()' $\underline{\underline{0}}$

$ $ $\underline{\underline{1}}$ 4

$$\boxed{\underline{\underline{C}} = \underline{\underline{U}}\ \underline{\underline{U}} = \underline{\underline{U}}} \qquad 7\ 58$$

, '), $

$ 4

$$\boxed{\underline{\underline{E}} = -\left(\underline{\underline{U}} - \underline{\underline{I}}\right)} \qquad 7\ \ 8$$

3 $ { Ī % Ī % Ī }% $ 0 % = 1
, $ + 3 { 3 % 3 % 3 }

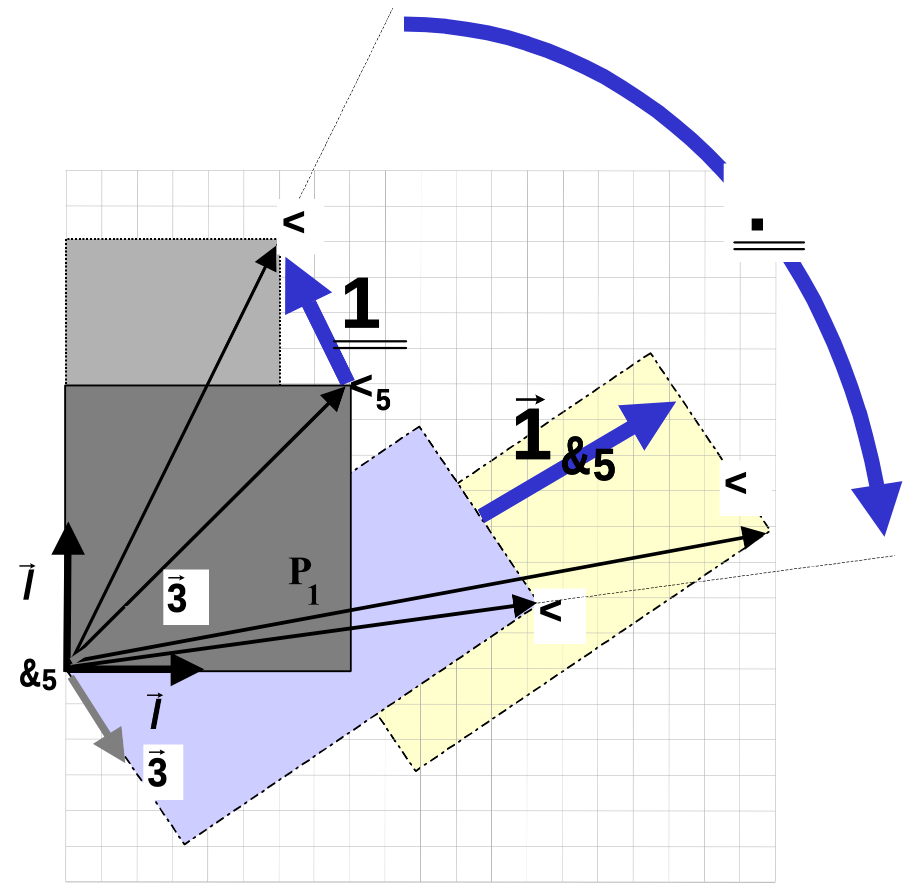


Figure 11: Signification géométrique de la décomposition RU.

0 &5 <5 $ $ #
$ $ = &5<5 , = # $ Φ
$ $ 3 $ $ = & < = E = = , # $
1 <5 < &5 $, # " $ &5< $
" $ &5<5 2 # " $ 3 Ī Ī # $
$ 1 , # $. < < &5 $, . %
" $ &5< $ " $ &5< %
$, " $ 3 Ī Ī " $

3 $ $\vec{3}$ $\vec{3}$ " $ $\vec{1}_{\&5}$ 7

8 &5 & < <

0 $ $ # " $ 3

$ $ " : $ " , $

4

$$\vec{\Phi} = \underline{\underline{.}}7\ 8\vec{=} + \vec{L}7\ 8$$

, , # *

$ - #$ " 4

$$\underline{\underline{E}} = \underline{\underline{.}} \rightarrow \underline{\underline{E}} = \underline{\underline{E}}^{-}$$

$$\underline{\underline{0}} = \underline{\underline{E}}\ \underline{\underline{E}} = _{=}$$

$$\underline{\underline{I}} = \underline{\underline{E}}\ \underline{\underline{E}} = _{=}$$

$$\underline{\underline{\ }} = \underline{\underline{E}}\ \underline{\underline{E}} = _{=}$$

VI DEFORMATION EN PETITE TRANSFORMATION

VI.1. Petites transformations, petites déformations, petits déplacements

" 0 $ () ' 0
'), $ $ Q 4

$$\underline{\underline{0}} = {}_{=} + \underline{\underline{\varepsilon}} + \underline{\underline{Q}}\ \underline{\underline{Q}}$$

$$_{=} = \underline{\underline{\varepsilon}} + -\underline{\underline{Q}}\ \underline{\underline{Q}}$$

6 $ Q 7Q ? = ∂ 1 @∂ = ? 8 " # 7$ 3
8% Q Q # 0 =
% ‖Q‖

$$\eta = \left\|\underline{\underline{Q}}\right\| = \sqrt{\sum_{\%\,=} Q}$$

; " #(# $ $
$; " $
$ " , $ $ (,
($ 7 $ % $
8 # #(Q&& 4 Q(
3 $

VI.2. Expression des tenseurs C et E sous l'hypothèse HPP

$$\underline{\underline{0}} = {}_{=} + \underline{\underline{\varepsilon}} + \underline{\underline{Q}}\ \underline{\underline{Q}} \Rightarrow \boxed{\underline{\underline{0}} \cong {}_{=} + \underline{\underline{\varepsilon}}} \qquad 7\ \ 8$$

$$_{=} = \underline{\underline{\varepsilon}} + -\underline{\underline{Q}}\ \underline{\underline{Q}} \Rightarrow \boxed{_{=} \cong \underline{\underline{\varepsilon}}} \qquad 7\ \ 8$$

, λ $ $\vec{\underline{\underline{=}}}$ $\lambda = \| \vec{} \| @ \| \vec{\underline{\underline{=}}} \|$ #

δ $ $\vec{\underline{\underline{=}}}$ $\lambda = \left(\| \vec{} \| - \| \vec{\underline{\underline{=}}} \| \right) @ \| \vec{\underline{\underline{=}}} \|$ "

4

$$\lambda = \frac{\| \vec{} \|}{\| \vec{\underline{\underline{=}}} \|} = \sqrt{\vec{5}\, \underline{\underline{0}}\, \vec{5}} \quad " \$ \ \underline{\underline{0}} = \ _{=} + _{=}$$

$$\$ \ \lambda = \sqrt{\ + \vec{5}\ _{=} \vec{5}}$$

- " $ δ = λ − 6 # (3 % 3 4

$\lambda = \sqrt{\ + \vec{5}\ _{=} \vec{5}} = \sqrt{\ + \vec{5}\, \underline{\underline{\varepsilon}}\, \vec{5}}$ $\vec{5}\, \underline{\underline{\varepsilon}}\, \vec{5}$ " # 0 $ $ *

% 4

$$\boxed{\lambda = \ + \vec{5}\, \underline{\underline{\varepsilon}}\, \vec{5}} \qquad 7 \quad 8$$

$$\boxed{\delta = \vec{5}\, \underline{\underline{\varepsilon}}\, \vec{5}} \qquad 7 \quad 8$$

0 $\vec{5}$ " $ 3 $\vec{l}$ 3 $ 0

" $ " $ $\underline{\underline{0}}$ $_{=}$ # (3 %

3 $ $ " $ $\underline{\underline{\varepsilon}}$

VI.3. Conditions de compatibilité

$ % $ * $

: * $ $ $

V W 6 # $ W I %

? " , $

$ $ $ $ * $

$ $ 0 $ $ $ 3

- " # $ $ 3 % # $ $

$ 0 # (3

$$\varepsilon_{?} = -\left(\frac{\partial \mathbf{1}}{\partial =_{?}} + \frac{\partial \mathbf{1}_{?}}{\partial =}\right)$$

, $ $ $

#$ " $ " 4

$$\mathbf{Q}_{?} = \frac{\partial \mathbf{1}}{\partial =_{?}} \quad \omega_{?} = -\left(\frac{\partial \mathbf{1}}{\partial =_{?}} - \frac{\partial \mathbf{1}_{?}}{\partial =}\right)$$

, " $ # $ $

% $ " $

$$\frac{\partial \omega_{?}}{\partial =_{M}} = -\left(\frac{\partial \mathbf{1}}{\partial =_{?} \partial =_{M}} - \frac{\partial \mathbf{1}_{?}}{\partial = \partial =_{M}}\right)$$

? ($\left(\partial \mathbf{1}_{M} @ \partial = \partial =_{?}\right) @$ % # $ #$

$$\frac{\partial \omega_{?}}{\partial =_{M}} = -\left(\frac{\partial \mathbf{1}}{\partial =_{?} \partial =_{M}} + \frac{\partial \mathbf{1}_{M}}{\partial = \partial =_{?}}\right) - -\left(\frac{\partial \mathbf{1}_{M}}{\partial = \partial =_{?}} + \frac{\partial \mathbf{1}_{?}}{\partial = \partial =_{M}}\right)$$

$ $ $ W " 3 -

" " # " K "

$ 4

$$\frac{\partial \omega_{?}}{\partial =_{M}} = \frac{\partial}{\partial =_{?}}\left\{-\left(\frac{\partial \mathbf{1}}{\partial =_{M}} + \frac{\partial \mathbf{1}_{M}}{\partial =}\right)\right\} - \frac{\partial}{\partial =}\left\{-\left(\frac{\partial \mathbf{1}_{M}}{\partial =_{?}} + \frac{\partial \mathbf{1}_{?}}{\partial =_{M}}\right)\right\} \qquad \frac{\partial \omega_{?}}{\partial =_{M}} = \frac{\partial \varepsilon_{M}}{\partial =_{?}} - \frac{\partial \varepsilon_{?M}}{\partial =}$$

6 $(\underline{\underline{\varepsilon}} = 5)$ % $\underline{\underline{\omega}}$ $

$ $\underline{\underline{Q}}$ $\underline{\underline{\omega}}$ 4

$$\underline{\underline{\varepsilon}} = 5 \Rightarrow \underline{\underline{\omega}} = \$\underline{\underline{\qquad\qquad}}\%\quad \underline{\underline{Q}} = \underline{\underline{\omega}}$$

, $ $ $ * $\vec{1}_5$ $

$$Q_? = \frac{\partial 1}{\partial =_?} = \omega_?(\ ) \Rightarrow 1 = \omega_?(\ ) =_? + 1_5$$

- % $ $ $ $ $ *

$

-" # $ $ 3 % # $ $

$$\boxed{\frac{\partial \omega_?}{\partial =_M} = \frac{\partial\ \varepsilon_M}{\partial =_?} - \frac{\partial\ \varepsilon_{?M}}{\partial =}} \qquad (7\ !8)$$

1 $ $ (3

$$\frac{\partial\ \omega_?}{\partial =_M \partial =} = \frac{\partial\ \omega_?}{\partial = \partial =_M} 7 \neq M8$$

, # $ $

$$\boxed{\frac{\partial\ \varepsilon_M}{\partial =_? \partial =} - \frac{\partial\ \varepsilon_{?M}}{\partial = \partial =_?} = \frac{\partial\ \varepsilon}{\partial =_? \partial =_M} - \frac{\partial\ \varepsilon_?}{\partial = \partial =_M}} \qquad (7\ F8)$$

0 (% ! * ,

$ (# 7 !8 3 ;

$ $\underline{\underline{\omega}}$ # $ K $\underline{\underline{\varepsilon}}$ - %

$ $\underline{\underline{Q}}$, #

$ * $ $ - (

$$\frac{\partial 1}{\partial = ?} = \varepsilon ? + \omega ?$$

3 X, " $\frac{\partial\ 1}{\partial = ?\partial =} = \frac{\partial\ 1}{\partial = \partial = ?}$ 7 $\neq$?8

$$\frac{\partial}{\partial =}\left(\varepsilon ? + \omega ?\right) = \frac{\partial}{\partial = ?}(\varepsilon\ + \omega\ )\ 7\ \neq\ ?8$$

; $ 3 # 7 !8 , Q * Q #

$ $ $ 3 , !

7 F8 " #$ 4

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial}{\partial =}\left\{\frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =} - \frac{\partial\ \varepsilon}{\partial =}\right\}$$

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial}{\partial =}\left\{\frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =} - \frac{\partial\ \varepsilon}{\partial =}\right\}$$

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial}{\partial =}\left\{\frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =} - \frac{\partial\ \varepsilon}{\partial =}\right\}$$

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =}$$

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =}$$

$$\frac{\partial\ \varepsilon}{\partial = \partial =} = \frac{\partial\ \varepsilon}{\partial =} + \frac{\partial\ \varepsilon}{\partial =}$$

7 G8

, $ 3 " $ %

(3 ' 33 $ # %

$ 3 #$ " 4

$$\underbrace{\frac{\partial}{\partial x_k \partial x_k}\varepsilon_{ij}}_{\Delta\varepsilon_{ij}} + \underbrace{\frac{\partial}{\partial x_i \partial x_j}\overbrace{\varepsilon_{kk}}^{trace(\underline{\underline{\varepsilon}})}}_{gr\vec{a}d\{gr\vec{a}d[trace(\underline{\underline{\varepsilon}})]\}} - \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\varepsilon_{ik}\right)}_{gr\vec{a}d[div(\underline{\underline{\varepsilon}})]} - \frac{\partial}{\partial x_i \partial x_k}\varepsilon_{jk} = 5$$

0 $ * # " ' 33 4

$$\Delta\left(\underline{\underline{\varepsilon}}\right) + gr\vec{a}d\left\{gr\vec{a}d\left[trace\left(\underline{\underline{\varepsilon}}\right)\right]\right\} = gr\vec{a}d\left[div\left(\underline{\underline{\varepsilon}}\right)\right] + \left\{gr\vec{a}d\left[div\left(\underline{\underline{\varepsilon}}\right)\right]\right\}^T$$

7 H8

VII VITESSES DE DEFORMATION

VII.1. Tenseur gradient des vitesses $\underline{\underline{,}}$

& 3 $ % * $ $ E

$ 7 8 " # $ 7 $$ $ 8 "

: $ " # &

$ % " , " $

$

$$" = \frac{\partial \vec{}}{\partial} \qquad 7\ 58$$

, " $ $ #

$ * $ % $

$ * " $ " , $ "

$ $ $ $

$$" = \frac{\partial "}{\partial \vec{}}\ \vec{} = \underline{\underline{,}}\ \vec{} \qquad \underline{\underline{,}} = \frac{\partial "}{\partial \vec{}} \qquad 7\ \ 8$$

$\underline{\underline{,}}$ " $ $

VII.2. Variation temporelle du tenseur gradient de la transformation $\dot{\underline{\underline{E}}} = \frac{\partial \underline{\underline{E}}}{\partial}$

$ % " $\underline{\underline{E}}$ %

$\vec{} = \underline{\underline{E}}\ \vec{=}$, $\underline{\underline{,}}$ # # $ " "

$ $, # $ $\underline{\underline{E}}$ # # $

" " $ $ 4

$$" = \underline{\underline{,}}\ \vec{} = \underline{\underline{,}}\ \underbrace{\underline{\underline{E}}\ \vec{=}}_{\vec{}}$$

; # $ " " : 3 $ * " # $

" $ → 4

$$" = \frac{\partial(\ \vec{}\ )}{\partial} = \frac{\partial(\underline{\underline{E}}\ \vec{=})}{\partial} = \frac{\partial(\underline{\underline{E}})}{\partial}\ \vec{=} + \underline{\underline{E}}\underbrace{\frac{\partial(\ \vec{=})}{\partial}}_{5} = \dot{\underline{\underline{E}}}\ \vec{=}$$

" $ 9 $ " $ $\vec{=}$

$ 3 6 $ $ % $\vec{=}$ $ "

0 $

$ " " $ $\vec{=}$ $ %

3 4

"

$$\boxed{\dot{\underline{\underline{E}}} = \underline{\underline{,}}\,\underline{\underline{E}}} \qquad 7\ 8$$

VII.3. Tenseur vitesse de déformation et vitesse de rotation

' $\underline{\underline{,}}$ '

#3 $

(($ " $\underline{\underline{,}}$ %

4

$$\boxed{\dot{\underline{\underline{\varepsilon}}} = -\left(\underline{\underline{,}} + \underline{\underline{,}}\right) = -\left(\left(\frac{\partial "}{\partial \vec{}}\right) + \left(\frac{\partial "}{\partial \vec{}}\right)\right)} \qquad 7\ 8$$

$$\boxed{\dot{\varepsilon}_{?} = -\left(\ _{?} + \ _{?}\right) = -\left(\frac{\partial "}{\partial\ _{?}} + \frac{\partial "_{?}}{\partial}\right)} \qquad 7\ 38$$

$$\boxed{\dot{\underline{\underline{\omega}}} = -\left(\underline{\underline{,}} - \underline{\underline{,}}\right) = -\left(\left(\frac{\partial "}{\partial \vec{}}\right) - \left(\frac{\partial "}{\partial \vec{}}\right)\right)} \qquad 7\ 8$$

$$\boxed{\dot{\omega}_{?} = -\left(\ _{?} - \ _{?}\right) = -\left(\frac{\partial "}{\partial\ _{?}} - \frac{\partial "_{?}}{\partial}\right)} \qquad 7\ 38$$