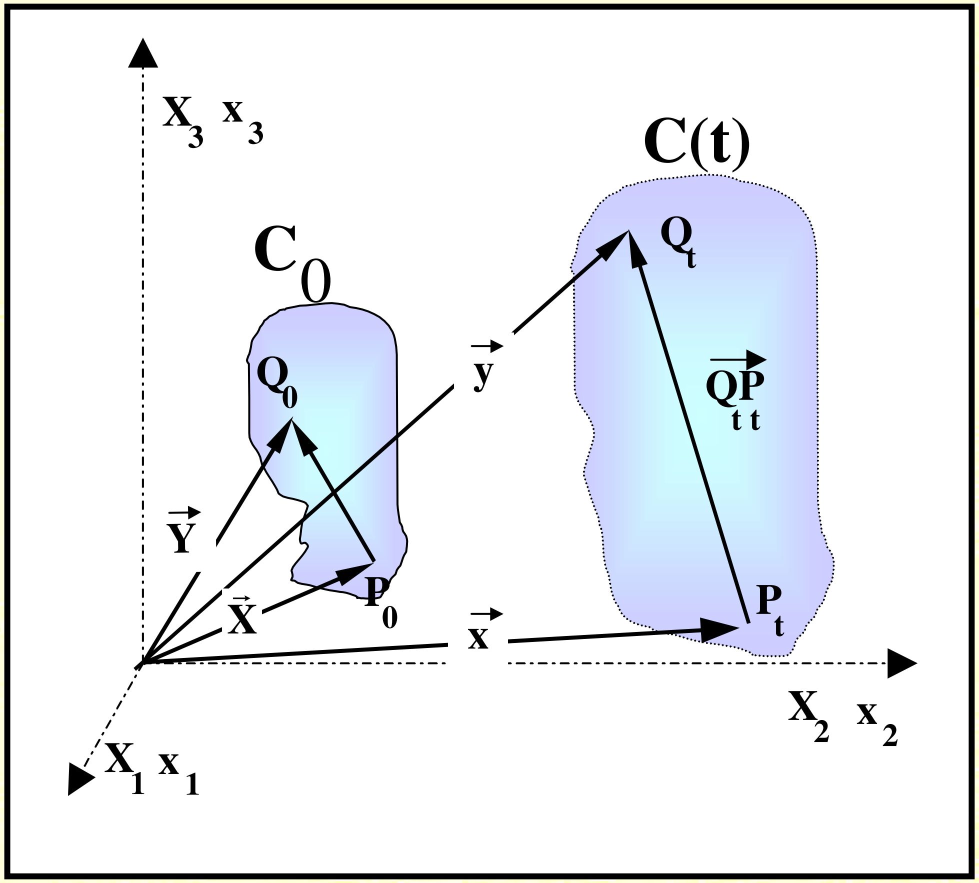
RAPPELS

I. Cinématique

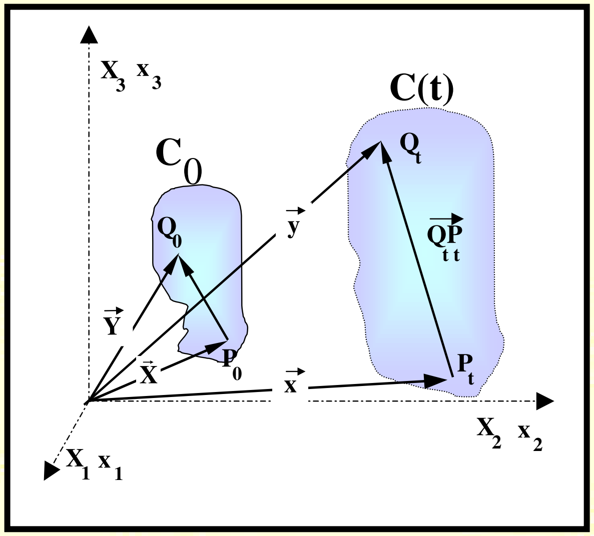


C_0 configuration initiale
description lagrangienne

$$\vec{x} = \vec{\Phi}(\vec{X}, t)$$

C(t) configuration actuelle
description eulérienne

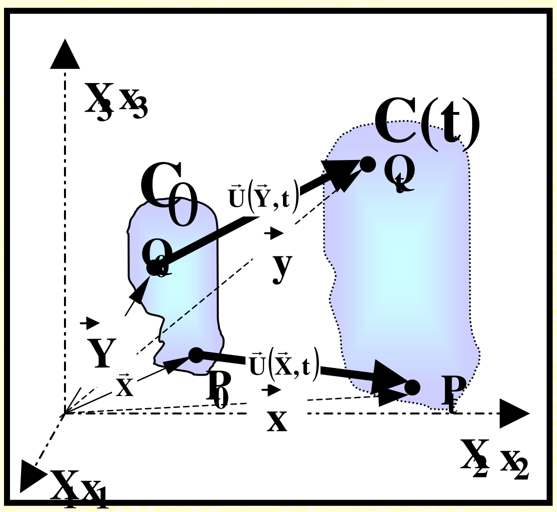
$$\vec{v}\left(\vec{x}, t\right) = \frac{d\vec{x}}{dt}$$



$$\underline{\underline{F}}(\vec{X},t) = \overrightarrow{grad}[\vec{\Phi}(\vec{X},t)]$$

$$d\vec{x} = \underline{\underline{F}}(\vec{X},t)d\vec{X}$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^{T}\,\underline{\underline{F}} - \underline{\underline{I}})$$

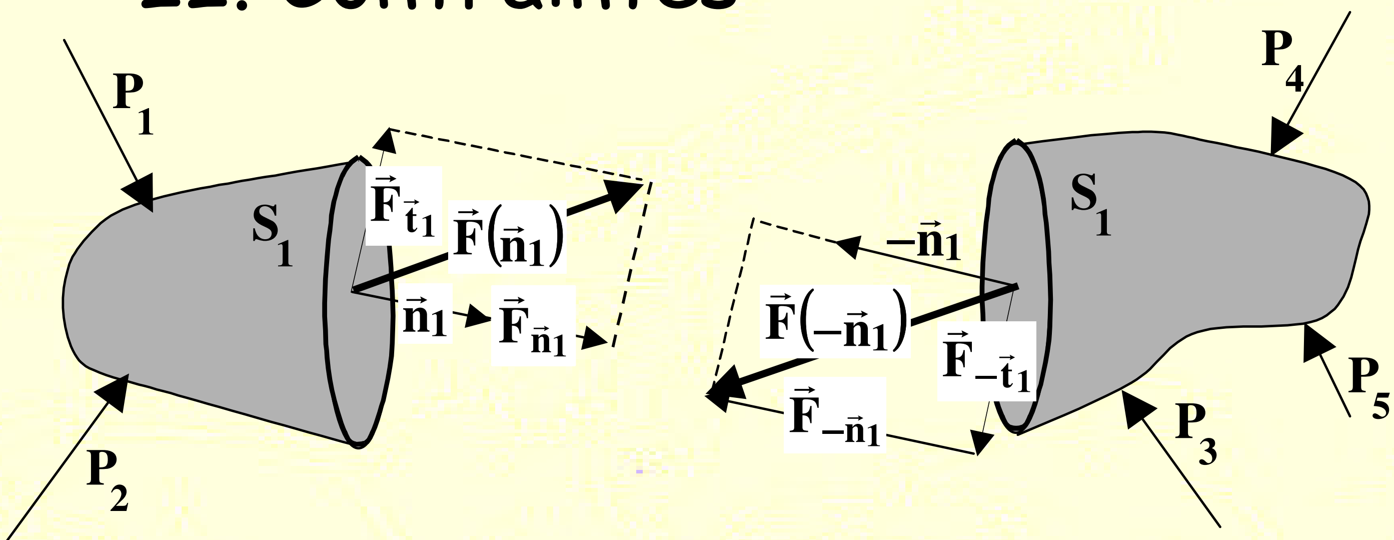


$$\underline{\underline{H}} = \overrightarrow{grad}\left[\vec{U}\left(\vec{X},t\right)\right]$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{H}} + \underline{\underline{H}}^{T}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$$

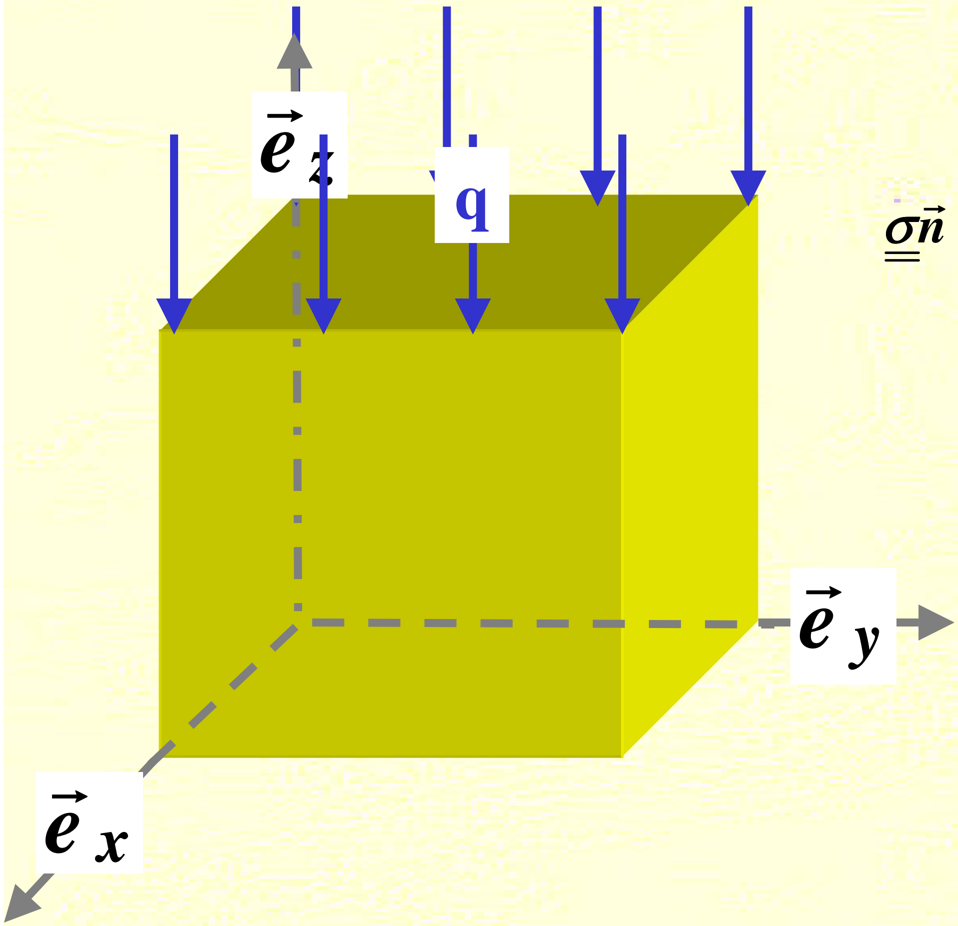
II. Contraintes



$$\vec{T} = \lim_{S \to 0} \frac{\vec{F}}{S} = T_{\vec{n}}\vec{n} + T_{\vec{t}}\vec{t}$$

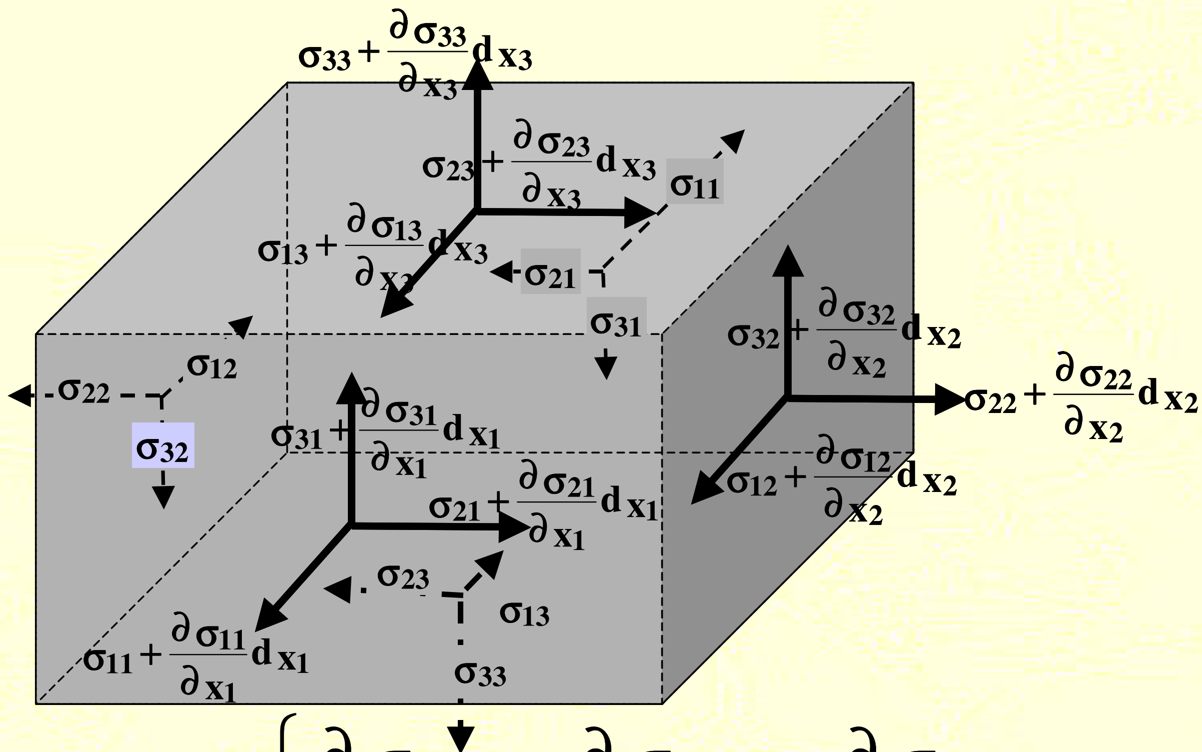
$$\vec{T}(\vec{n}) = \underline{\underline{\sigma}}\ \vec{n} \Leftrightarrow \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\vec{T}(\vec{n}) = \underline{\underline{\sigma}}\ \vec{n} = \vec{T}^{imposé}(\vec{n}) \text{ sur } S_\sigma$$



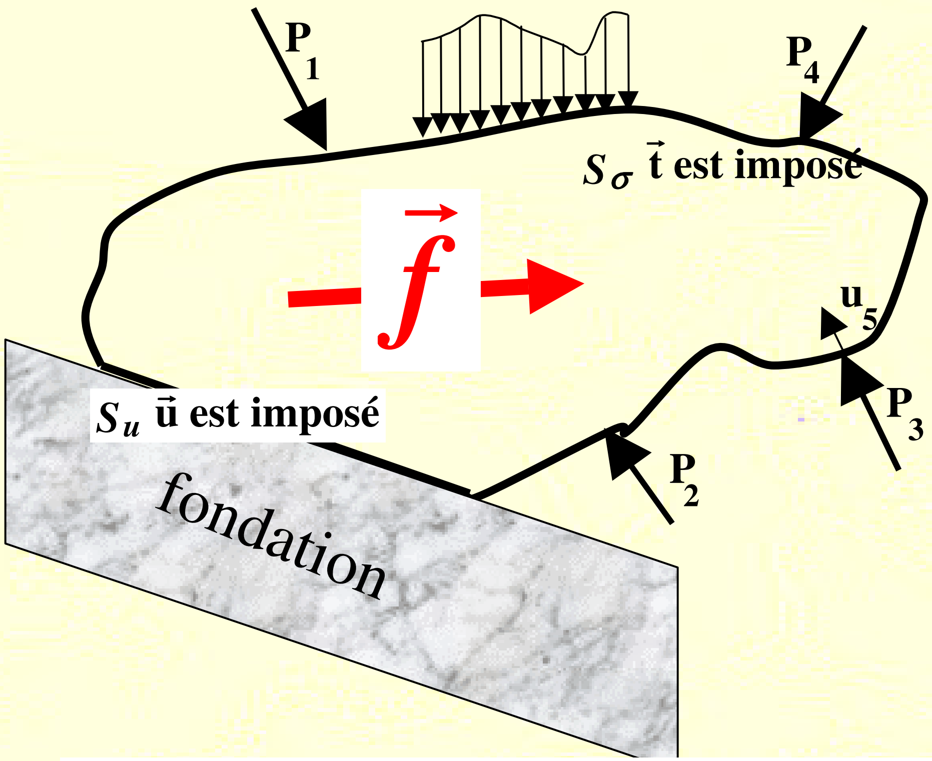
$$\underline{\underline{\sigma}}\vec{n} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q \end{bmatrix}$$

$$\begin{cases} \sigma_{xz} = 0 \\ \sigma_{yz} = 0 \\ \sigma_{zz} = -q \end{cases} \quad \begin{cases} \sigma_{xx} = ? \\ \sigma_{xy} = ? \\ \sigma_{yy} = ? \end{cases}$$



$$\vec{div}\left(\underline{\underline{\sigma}}\right) = 0 \begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} = 0 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \end{cases}$$

III. Equations de bilan



$\vec{f} = \rho\vec{g}$ par exemple

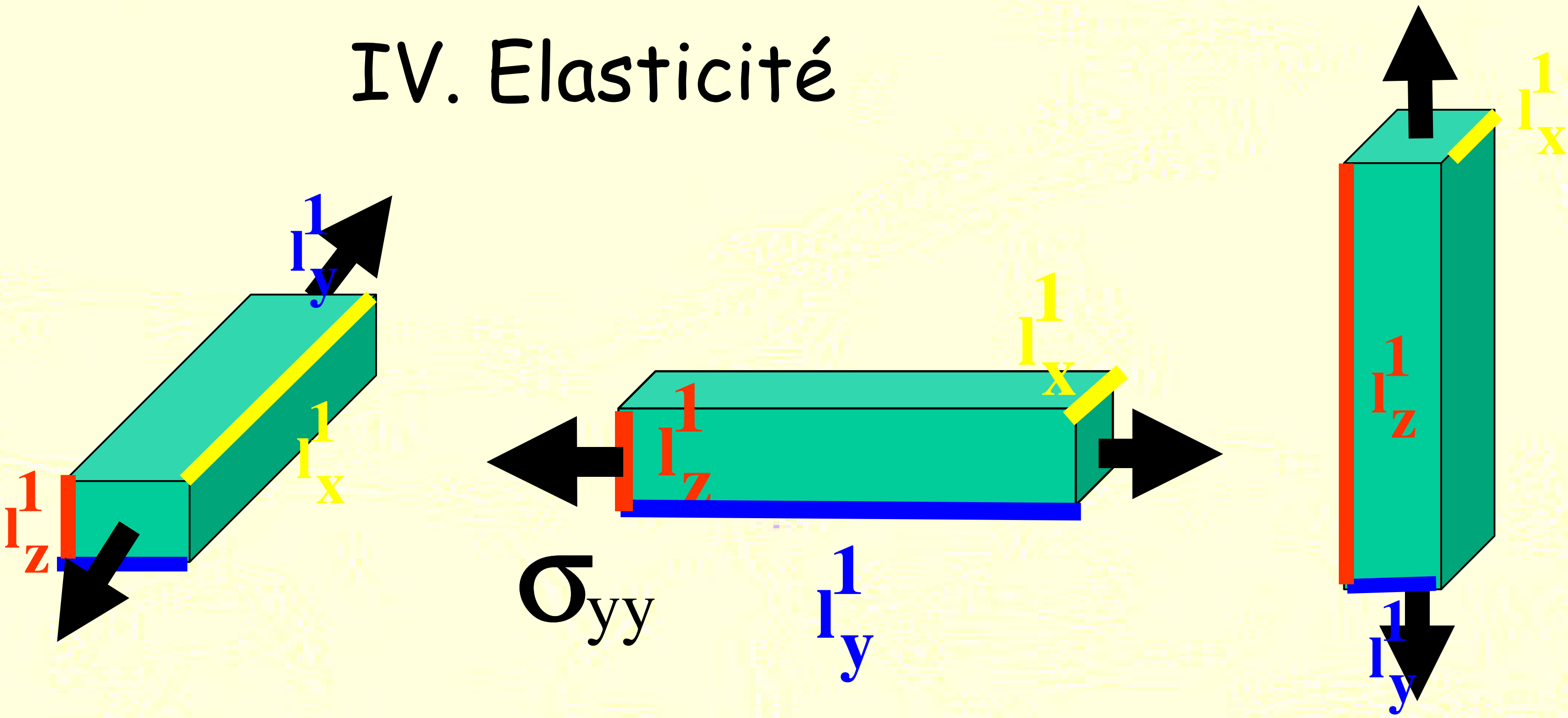
$$\int_{\Omega} \vec{f}dv + \int_{S_\sigma+S_u} \vec{t}dS = \int_{\Omega} \rho\vec{\gamma}dv$$

$$\underbrace{\int_{\Omega} \left\{ \vec{f} + div\left(\underline{\underline{\sigma}}\right) - \rho\vec{\gamma} \right\} dv = 0}$$

$$\Updownarrow$$

$$\vec{f} + div\left(\underline{\underline{\sigma}}\right) - \rho\vec{\gamma} = 0 \text{ dans } \Omega$$

IV. Elasticité



$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{yy} = -\nu \varepsilon_{xx} = -\nu \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{zz} = -\nu \varepsilon_{xx} = -\nu \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{xx} = -\nu \varepsilon_{yy} = -\nu \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{zz} = -\nu \varepsilon_{yy} = -\nu \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{xx} = -\nu \varepsilon_{zz}$$

$$\varepsilon_{yy} = -\nu \varepsilon_{zz}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E}$$

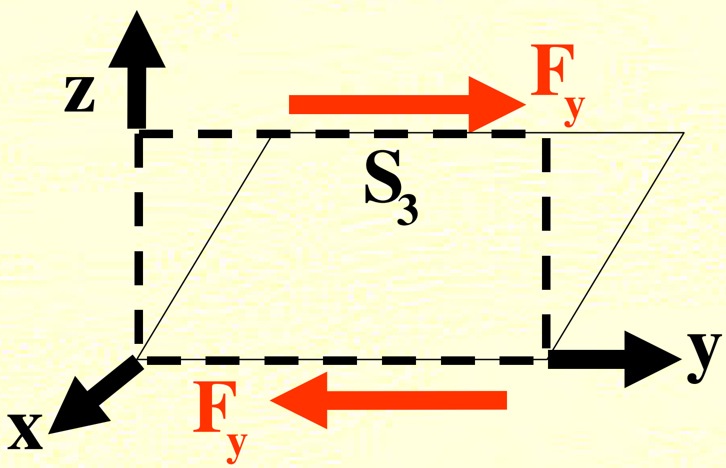
Traction le long de trois axes orthogonaux

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E}$$

Sollicitation en cisaillement



$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G}$$

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{2G}$$

$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G}$$

$$G = \frac{E}{2(1+\nu)}$$

$$\begin{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} \\ \begin{pmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{pmatrix} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 1+\upsilon & 0 & 0 \\ 0 & 1+\upsilon & 0 \\ 0 & 0 & 1+\upsilon \end{pmatrix} \end{bmatrix} * \begin{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix} \\ \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} \end{bmatrix}$$

La loi de Hooke complète pour un matériau anisotrope

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$$

$$\sigma_{ij} = \sigma_{ji}$$

$$\varepsilon_{kl} = \varepsilon_{lk}$$

$$\left.\right\} \quad L_{ijkl} = L_{jikl}, \quad L_{ijkl} = L_{ijkl}$$

IV. Energie de déformation

$$dW = \int_V \sigma_{ij} \Delta\varepsilon_{ij} dv \qquad W = \int_V \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} dv$$

$$W_{vol} = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$$

$$W_{vol} = \int_0^{\varepsilon} \vec{\sigma}^T d\vec{\varepsilon}$$

IV. L’énergie de déformation

IV.1. Expression de l’énergie en fonction des contraintes et des déformations

$$W_{vol} = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$$

$$\varepsilon_{ij} = M_{ijkl}\sigma_{kl}$$

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$$

$$\rightarrow W_{vol} = \frac{1}{2}\underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{M}}}} : \underline{\underline{\sigma}}$$

$$= \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{L}}}} : \underline{\underline{\sigma}}$$

IV. L'énergie de déformation

IV.1. Propriétés de M et L

$$W_{vol} = \frac{1}{2}\vec{\sigma}^T M \vec{\sigma} \geq 0 \rightarrow M \text{ est défini postif}$$

$$W_{vol} = \frac{1}{2}\vec{\varepsilon}^T L \vec{\varepsilon} \geq 0 \rightarrow L \text{ est défini postif}$$

Chapitre V : Méthodes semi-inverses

I. Bilan

I.1. Nombre d’inconnus

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Vecteur déplacement $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ → **3**

I. Bilan

I.1. Nombre d'inconnus

Vecteur déplacement $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ → **3**

Tenseur petites déformations $\vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{bmatrix}$ $\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \sigma_{22} & \sigma_{23} \\ \varepsilon_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$ → **6**

I. Bilan

I.1. Nombre d'inconnus

Vecteur déplacement $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ → **3**

Tenseur petites déformations $\vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \\ \varepsilon_{xy} \end{bmatrix}$ $\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \sigma_{22} & \sigma_{23} \\ \varepsilon_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$ → **6**

Tenseur contraintes de Cauchy $\vec{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{bmatrix}$ $\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$ → **6**

I. Bilan

I.2. Nombre d'équations en volume

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Équilibre : 3 équations scalaires

$$\partial \sigma_{ij} / \partial x_j + f_i = 0$$

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I.2. Nombre d'équations en volume

Équilibre : 3 équations scalaires

$$\partial\sigma_{ij}/\partial x_j + f_i = 0$$

Compatibilité : 6 équations scalaires

$$\frac{\partial^2}{\partial x_k \partial x_k}\varepsilon_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}\varepsilon_{kk} - \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\varepsilon_{ik}\right) - \frac{\partial^2}{\partial x_i \partial x_k}\varepsilon_{jk} = 0$$

I. Bilan

I.2. Nombre d'équations en volume

Équilibre : 3 équations scalaires

$$\partial \sigma_{ij} / \partial x_j + f_i = 0$$

Compatibilité : 6 équations scalaires

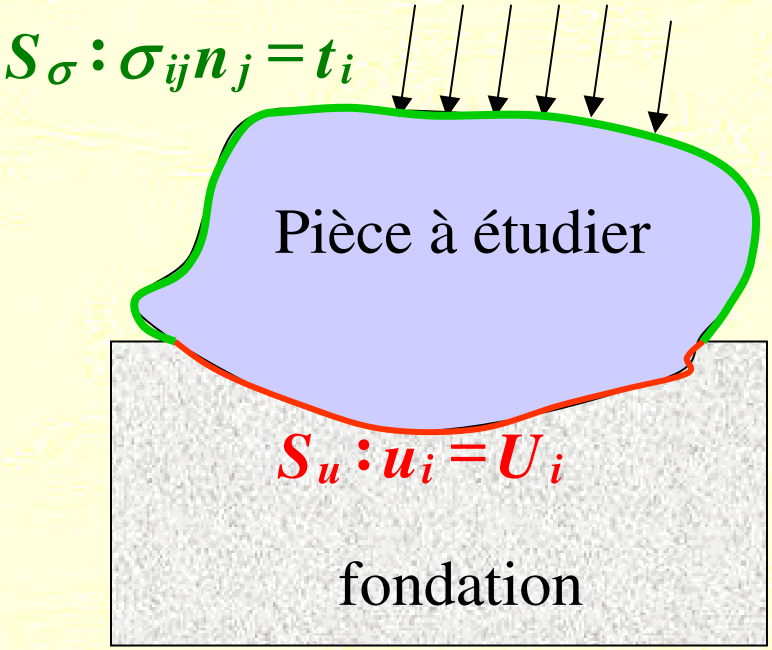
$$\frac{\partial^2}{\partial x_k \partial x_k} \varepsilon_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \varepsilon_{kk} - \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_k} \varepsilon_{ik} \right) - \frac{\partial^2}{\partial x_i \partial x_k} \varepsilon_{jk} = 0$$

Lois de Hooke : 6 équations scalaires

$$\varepsilon_{ij} = \frac{1}{E} \left\{ (1+\nu) \sigma_{ij} - \upsilon \sigma_{ll} \delta_{ij} \right\}$$

I. Bilan

I.2. Nombre de conditions limites



3 équations scalaires

$$\sigma_{ij} n_j = t_i \text{ sur } S_\sigma$$

ou

$$u_i = U_i \text{ sur } S_u$$

II. Equations fondamentales de l'élasticité

II.1. Démarche

15 équations différentielles du premier ordre

équilibre : $\partial \sigma_{ij} / \partial x_j + f_i = 0$

compatibilité : $\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} \right)$

Hooke : $\varepsilon_{ij} = \dfrac{1}{E}\left\{ (1+\nu)\sigma_{ij} - \nu \sigma_{ll} \delta_{ij} \right\}$

↓

3 équations différentielles scalaires du second ordre

sur $\underline{\underline{\sigma}}$ (équations de Beltrami-Mitchel)

ou

sur $\vec{u}$ (équations de Navier)

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.1. Cas général

$$\left.\begin{array}{l} \Delta\varepsilon_{ij} + gr\vec{a}d\left\{gr\vec{a}d\left[trace\left(\underline{\underline{\varepsilon}}\right)\right]\right\} \\ -gr\vec{a}d\left[div\left(\underline{\underline{\varepsilon}}\right)\right] - gr\vec{a}d\left[div\left(\underline{\underline{\varepsilon}}\right)\right]^T = 0 \\ \underline{\underline{\varepsilon}} = \dfrac{1}{E}\left\{\left(1+\nu\right)\underline{\underline{\sigma}} - \upsilon\, trace\left(\underline{\underline{\sigma}}\right)\underline{\underline{I}}\right\} \\ \overrightarrow{div}\left(\underline{\underline{\sigma}}\right) + \vec{f} = 0 \end{array}\right\} \rightarrow \begin{array}{l} \left(1+\nu\right)\Delta\underline{\underline{\sigma}} \\ +\overrightarrow{grad}\left\{\overrightarrow{grad}\left[trace\left(\underline{\underline{\sigma}}\right)\right]\right\} \\ +\left(1+\nu\right)\left\{\begin{array}{l} \overrightarrow{grad}\left[\vec{f}\right] \\ +\left\{\overrightarrow{grad}\left[\vec{f}\right]\right\}^T \end{array}\right\} \\ +div\left[\vec{f}\right]\underline{\underline{I}} = 0 \end{array}$$

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.1. Cas général

$$\overbrace{(1+\nu)\Delta\underline{\underline{\sigma}} + \overrightarrow{grad}\left\{\overrightarrow{grad}\left[\overbrace{trace\left(\underline{\underline{\sigma}}\right)}^{\text{role particulier}}\right]\right\}}^{\textit{dérivées du second ordre de } \underline{\underline{\sigma}}}$$

$$\underbrace{+(1+\nu)\left\{\overrightarrow{grad}\left[\vec{f}\right] + \left\{\overrightarrow{grad}\left[\vec{f}\right]\right\}^{T}\right\} + div\left[\vec{f}\right]\underline{\underline{I}}}_{\textit{dérivées du premier ordre de } \vec{f}} = 0$$

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.2. Forces de volume nulles ou constantes

$$(1+\nu)\Delta\underline{\underline{\sigma}} + \overrightarrow{grad}\left\{\overrightarrow{grad}\,trace\left(\underline{\underline{\sigma}}\right)\right\} = 0$$

$$\Delta\sigma_{mm} = \Delta\left[trace\left(\underline{\underline{\sigma}}\right)\right]0$$

$$\Delta\Delta\sigma_{ij} = \Delta\Delta\underline{\underline{\sigma}} = 0$$

II. Equations fondamentales de l'élasticité

II.2. Equation de Beltrami-Mitchell

II.2.3. Application : contraintes planes ou déformations planes

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{l} div\left(\underline{\underline{\sigma}}\right) = 0 \rightarrow \begin{cases} \sigma_{xx} = \dfrac{\partial \varphi^2}{\partial y^2} \\ \sigma_{xy} = -\dfrac{\partial \varphi^2}{\partial x \partial y} \\ \sigma_{yy} = \dfrac{\partial \varphi^2}{\partial x^2} \end{cases} \\ \underline{\underline{\varepsilon}} = \dfrac{1}{2}\left\{ grad\left(\vec{u}\right) + \left[grad\left(\vec{u}\right)\right]^T \right\} \\ \underline{\underline{\sigma}} = 2G\left\{ \underline{\underline{\varepsilon}} + \dfrac{\upsilon}{1-2\upsilon} trace\left(\underline{\underline{\varepsilon}}\right) \underline{\underline{I}} \right\} \end{array}\right\} \rightarrow \Delta\Delta\varphi = 0$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas général

$$\left.\begin{array}{l} div\left(\underline{\underline{\sigma}}\right)=\vec{f} \\ \underline{\underline{\varepsilon}}=\dfrac{1}{2}\left\{\begin{array}{l} grad\left(\vec{u}\right) \\ +\left[grad\left(\vec{u}\right)\right]^T \end{array}\right\} \\ \left\{\begin{array}{l} \underline{\underline{\sigma}}=2G\left\{\underline{\underline{\varepsilon}}+\dfrac{\nu}{1-2\nu}\theta\underline{\underline{I}}\right\} \\ \theta=trace\left(\underline{\underline{\varepsilon}}\right) \end{array}\right. \end{array}\right\}\rightarrow \overbrace{\Delta\vec{u}+\underbrace{\left(\frac{1}{1-2\nu}\right)}_{\text{comportement du matériau}}\overrightarrow{grad}\left[div\left(\vec{u}\right)\right]}^{\text{dépend du champ des déplacements}}$$

$$+\frac{\overbrace{\vec{f}}^{\text{forces volumiques}}}{\underbrace{G}_{\text{comportement du matériau}}}=0$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas particuliers

a) Forces de volume nulles

$$\Delta\vec{u} + \underbrace{\left(\frac{1}{1-2\nu}\right)}_{\substack{\textit{comportement}\\ \textit{du matériau}}}\overrightarrow{grad}\left[div\left(\vec{u}\right)\right] = 0$$

$$\vec{u} = \frac{1}{2G}\overrightarrow{grad}\left[\vec{X}.\vec{\Phi}\right] + \frac{1-\upsilon}{G}\vec{\Phi}$$

$$\Delta\vec{\Phi} = 0$$

$$\vec{\Phi} = \textit{la nouvelle inconnue}$$

II. Equations fondamentales de l'élasticité

II.3. Equation de Navier

II.3.1. Cas particuliers

a) Forces de volume dérivant d'un potentiel

$$\Delta\vec{u} + \left(\frac{1}{1-2\nu}\right)\overrightarrow{grad}\left[div\left(\vec{u}\right)\right] + \frac{\vec{f}}{G} = 0$$

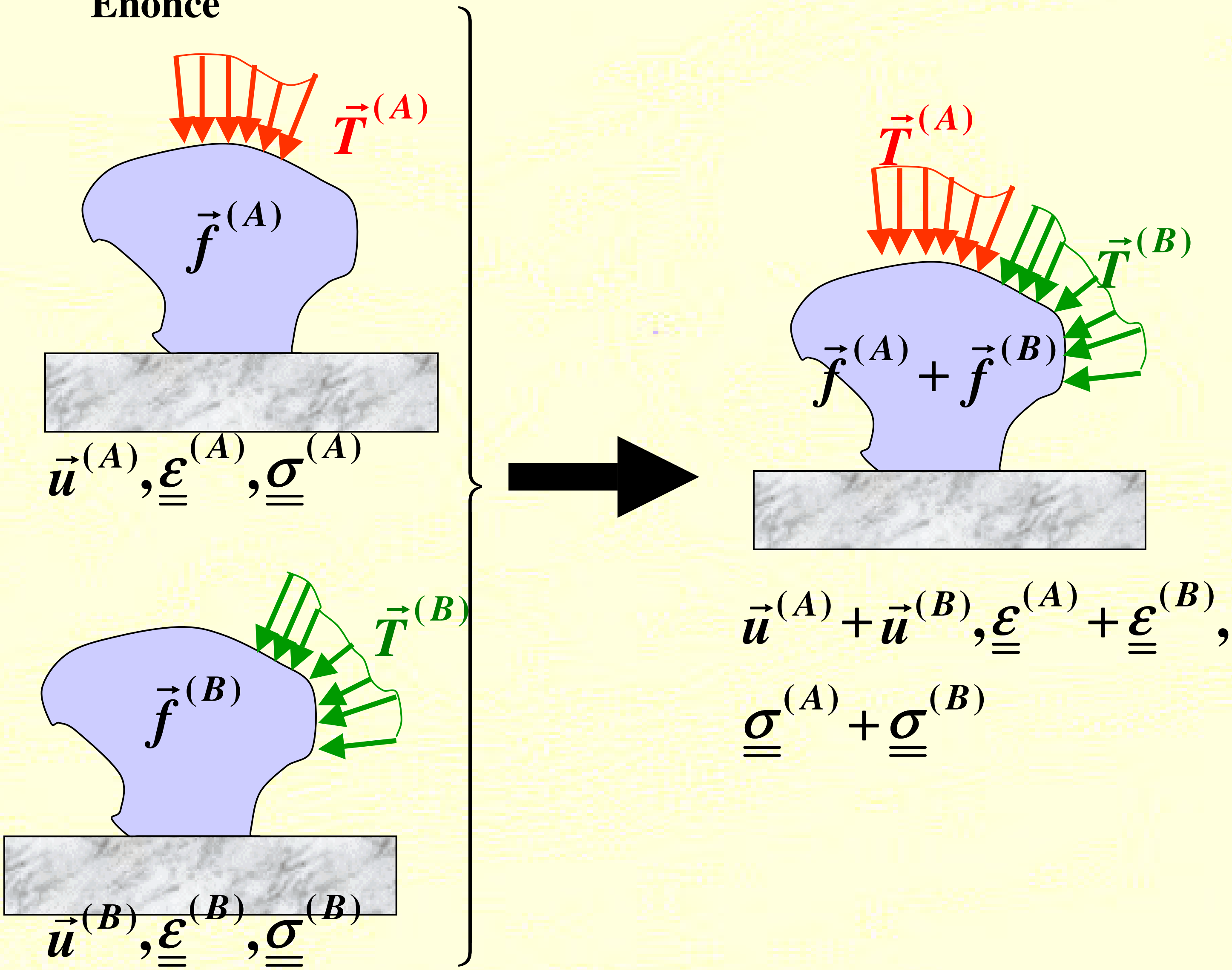
$$\vec{f} = \overrightarrow{grad}\left(V\right)$$

$$\Delta\vec{u} + \frac{1}{1-2\nu}\overrightarrow{grad}\left[div\left(\vec{u}\right)\right] + \frac{\overrightarrow{grad}\left(V\right)}{G} = 0$$

III. Principe de superposition et unicité de la solution

III.1. Principe de superposition

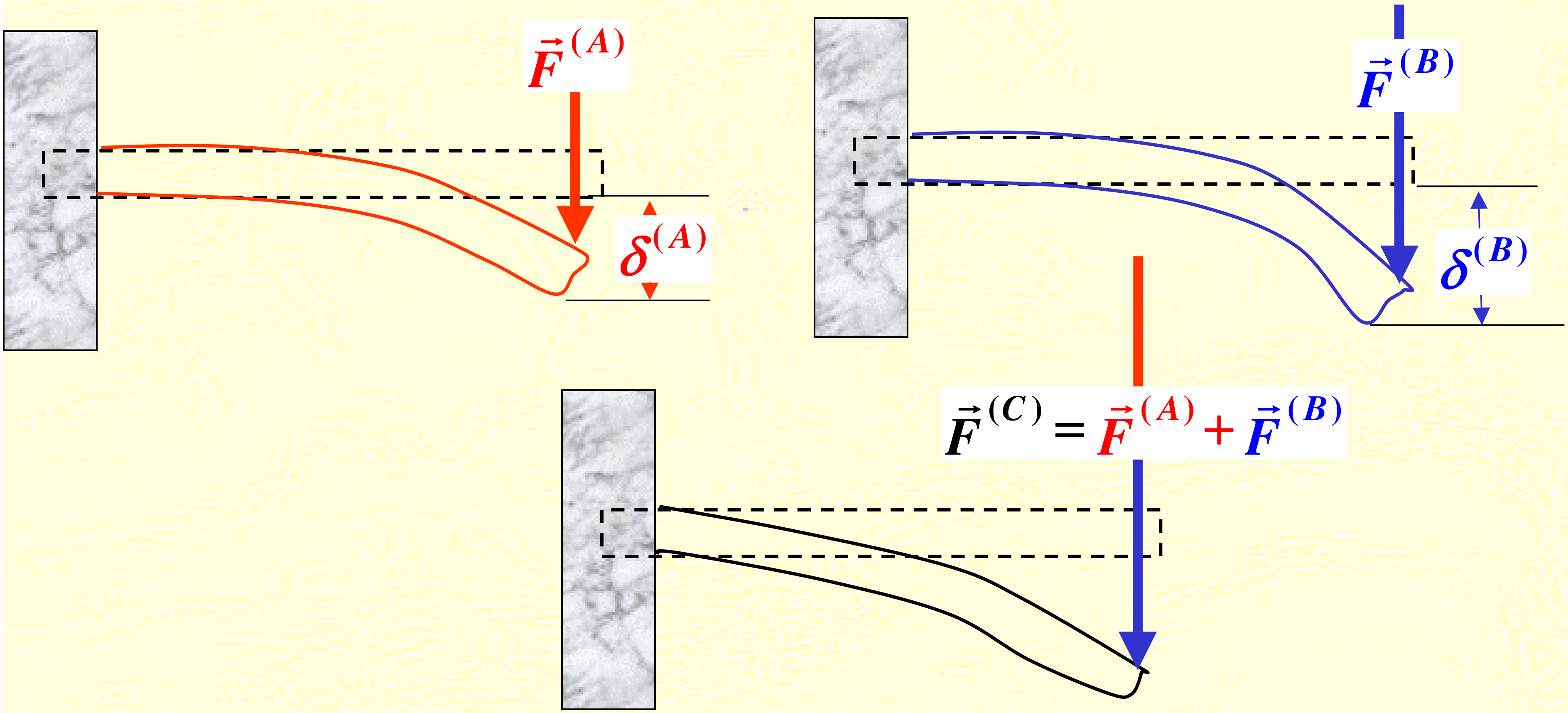
Enoncé



III. Principe de superposition et unicité de la solution

III.1. Principe de superposition

limitation : hypothèse des petites perturbations

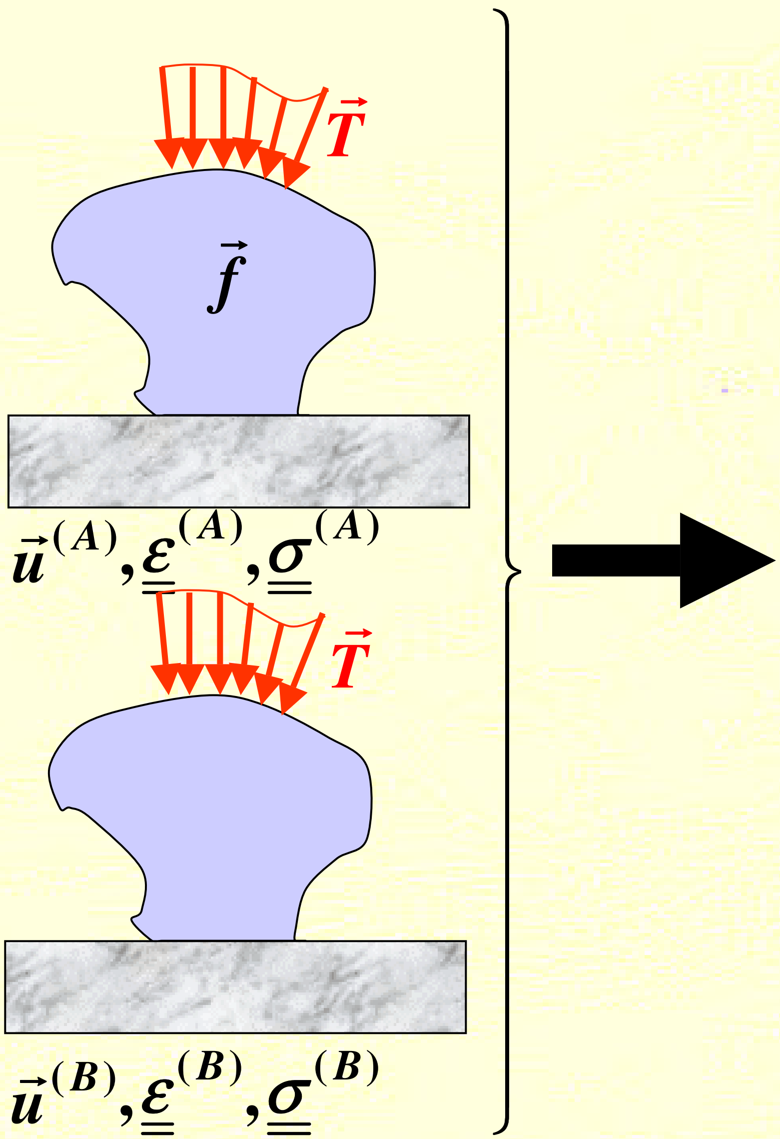


$$\vec{F}^{(C)} = \vec{F}^{(A)} + \vec{F}^{(B)}$$

$$\delta^{(C)} < \delta^{(A)} + \delta^{(B)}$$

III. Principe de superposition et unicité de la solution

III.2. Unicité de la solution



$$\vec{u}^{(A)} - \vec{u}^{(B)} = 0?$$

$$\underline{\underline{\varepsilon}}^{(A)} - \underline{\underline{\varepsilon}}^{(B)} = 0?$$

$$\underline{\underline{\sigma}}^{(A)} - \underline{\underline{\sigma}}^{(B)} = 0?$$

III. Principe de superposition et unicité de la solution

III.2. Unicité de la solution

$$\left.\begin{array}{l}\vec{u}\,'=\vec{u}^{(A)}-\vec{u}^{(B)}\\ \underline{\underline{\varepsilon}}\,'=\underline{\underline{\varepsilon}}^{(A)}-\underline{\underline{\varepsilon}}^{(B)}\\ \underline{\underline{\sigma}}\,'=\underline{\underline{\sigma}}^{(A)}-\underline{\underline{\sigma}}^{(B)}\end{array}\right\}\ \text{est solution du problème}\ \begin{cases}\vec{T}\,'=\vec{T}-\vec{T}=0\\ \vec{f}\,'=\vec{f}-\vec{f}=0\end{cases}$$

$$\int_V \vec{\sigma}\,'^T\,\underline{\underline{M}}\,\vec{\sigma}\,'\,dv=\int_S \vec{T}\,'^T\vec{u}\,'\,dS=0$$

$$\left.\begin{array}{l}\displaystyle\int_V \underbrace{\left[\vec{\sigma}^A-\vec{\sigma}^B\right]}_{\vec{\sigma}\,'}^T\,\underline{\underline{M}}\underbrace{\left[\vec{\sigma}^A-\vec{\sigma}^B\right]}_{\vec{\sigma}\,'}dv=0\\ \underline{\underline{M}}\ \text{est défini positif}\end{array}\right\}\rightarrow\vec{\sigma}^A-\vec{\sigma}^B=0$$