3D plasticity

- **3D viscoplasticity**
- **3D plasticity**
- **Perfectly plastic material**
- **Direction of plastic flow with various criteria**
- **Prandtl-Reuss, Hencky-Mises, Prager rules**

–Write 3D equations for inelastic behavior–

3D plasticity and viscoplasticity

- Strain partition

$$\underset{\sim}{\varepsilon}^e = \underset{\approx}{\Lambda}^{-1} : (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_I)$$

$$\underset{\sim}{\varepsilon}^{th} = (T - T_I)\,\underset{\sim}{\alpha}$$

$$\underset{\sim}{\varepsilon} = \underset{\approx}{\Lambda}^{-1} : (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_I) + \underset{\sim}{\varepsilon}^{th} + \underset{\sim}{\varepsilon}^{p} + \underset{\sim}{\varepsilon}^{vp}$$

- Criterion

$$f$$

- Flow rule

$$\dot{\underset{\sim}{\varepsilon}}^p = ...$$

- Hardening rule

$$\dot{Y}_I = ...$$

Formulation of viscoplastic constitutive equations

The easiest way of writing a viscoplastic model is to define a *viscoplastic potential*, Φ, depending on stress and hardening variables. A *standard* model will then be characterized using the yield function f to define Φ, and deriving viscoplastic strain rate and hardening rate from Φ, $\Phi := \Phi(f(\underset{\sim}{\boldsymbol{\sigma}}, Y_I))$.

- Viscoplastic strain rate:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{vp} = \frac{\partial \Phi}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$$

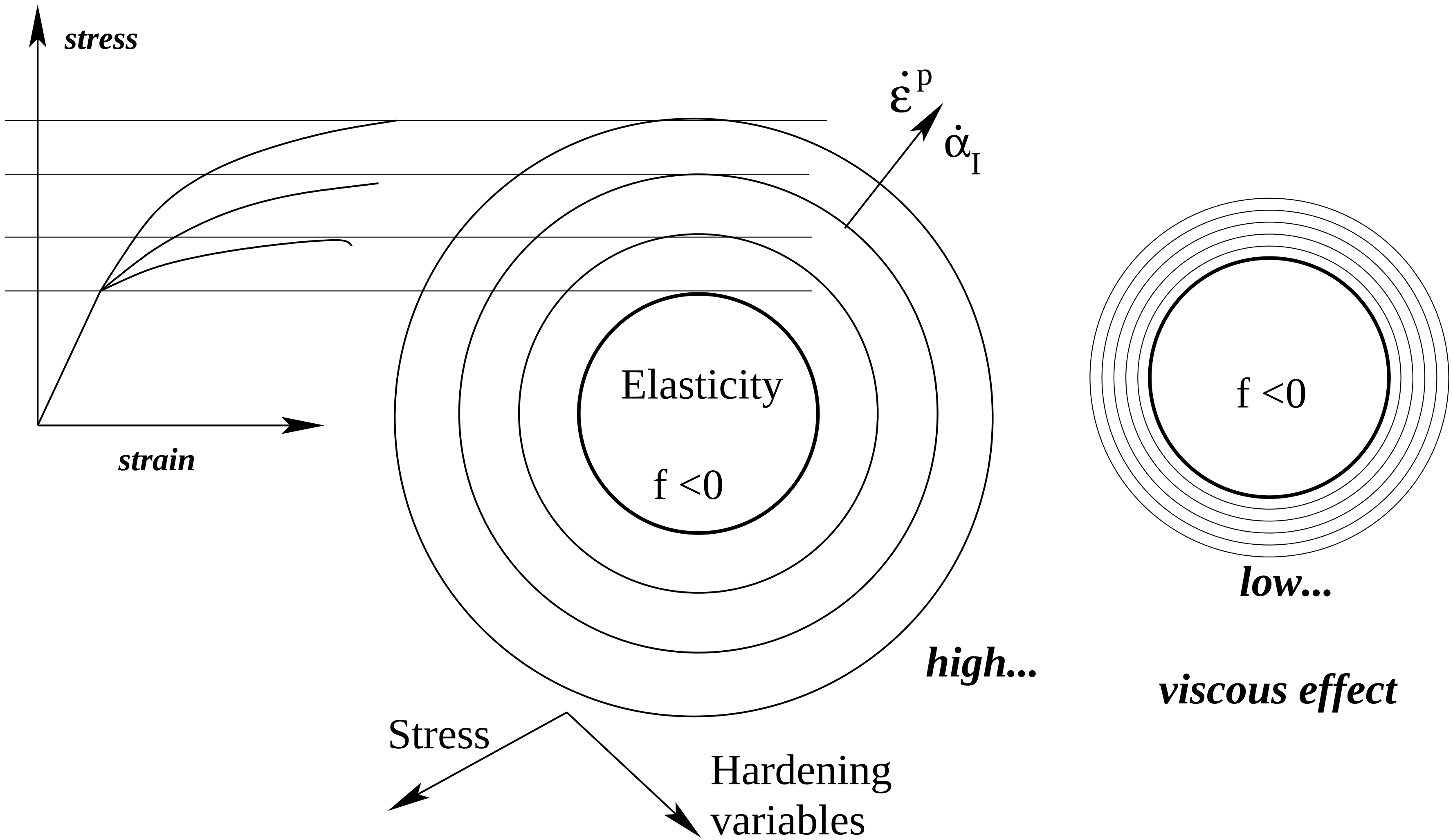
- State variable rate:

$$\dot{\alpha}_I = -\frac{\partial \Phi}{\partial Y_I}$$

Introducing $\dot{v} = \dfrac{\partial \Phi}{\partial f}$, $\underset{\sim}{\boldsymbol{n}} = \partial f / \partial \underset{\sim}{\boldsymbol{\sigma}}$, and $M_I = \partial f / \partial Y_I$

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^{vp} = \dot{v}\, \underset{\sim}{\boldsymbol{n}} \qquad \dot{\alpha}_I = -\dot{v}\, M_I$$

Viscoplastic potential, standard model



$$\dot{\underset{\sim}{\varepsilon}}^{vp} = \dot{v}\underset{\sim}{\boldsymbol{n}} \qquad \dot{\alpha}_I = -\dot{v}M_I$$

Examples of simple viscoplastic models

- Norton rule and von Mises criterion $f = J(\underset{\sim}{\boldsymbol{\sigma}})$, and :

$$\Phi = \frac{K}{n+1} \left(\frac{J(\underset{\sim}{\boldsymbol{\sigma}})}{K} \right)^{n+1}$$

$$\underset{\sim}{\dot{\boldsymbol{\varepsilon}}}^{vp} = \left(\frac{J}{K} \right)^{n} \frac{\partial J}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$$

$$\frac{\partial J}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{\partial J}{\partial \underset{\sim}{\boldsymbol{s}}} : \frac{\partial \underset{\sim}{\boldsymbol{s}}}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{3}{2} \frac{\underset{\sim}{\boldsymbol{s}}}{J} : (\underset{\approx}{\boldsymbol{I}} - \frac{1}{3} \underset{\sim}{\boldsymbol{I}} \otimes \underset{\sim}{\boldsymbol{I}}) = \frac{3}{2} \frac{\underset{\sim}{\boldsymbol{s}}}{J}$$

The elastic domain is reduced to one point.

- Bingham model:

$$\Phi = \frac{1}{2} \left(\frac{J(\underset{\sim}{\boldsymbol{\sigma}}) - \sigma_y}{\eta} \right)^2$$

NOTE: partial derivative of $\underset{\sim}{\sigma}$ with respect to $\underset{\sim}{s}$

- Tensor $\underset{\approx}{\boldsymbol{J}} = \underset{\approx}{\boldsymbol{I}} - \frac{1}{3}\underset{\sim}{\boldsymbol{I}} \otimes \underset{\sim}{\boldsymbol{I}}$

$$\underset{\sim}{\boldsymbol{s}} = \underset{\approx}{\boldsymbol{J}} : \underset{\sim}{\boldsymbol{\sigma}}$$

- Index notation:

$$J_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}$$

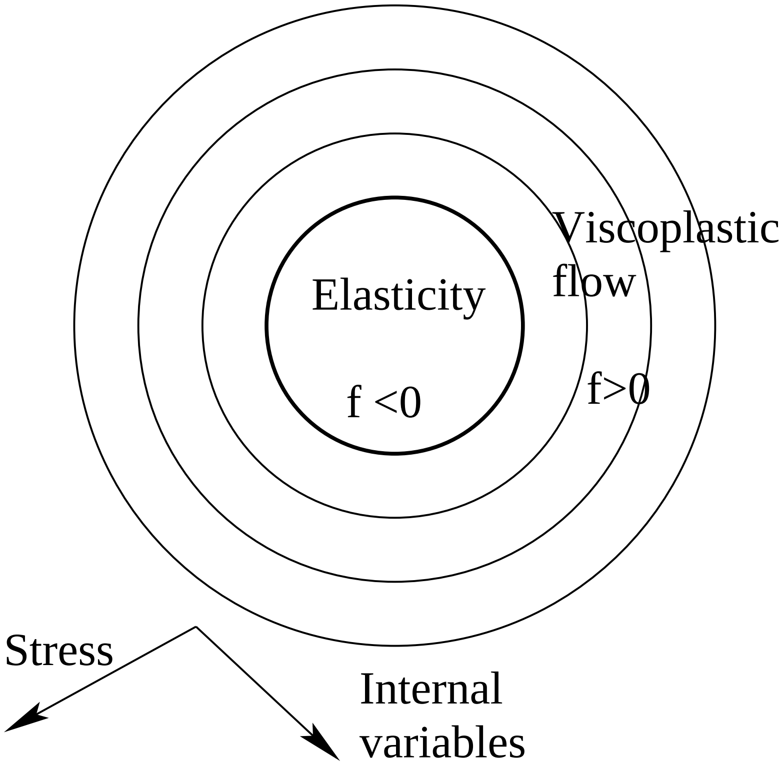
- Other solution:

$$J^2 = \frac{3}{2}s_{ij}s_{ij} \qquad \text{then} \qquad 2JdJ = 3s_{ij}d\sigma_{ij}$$

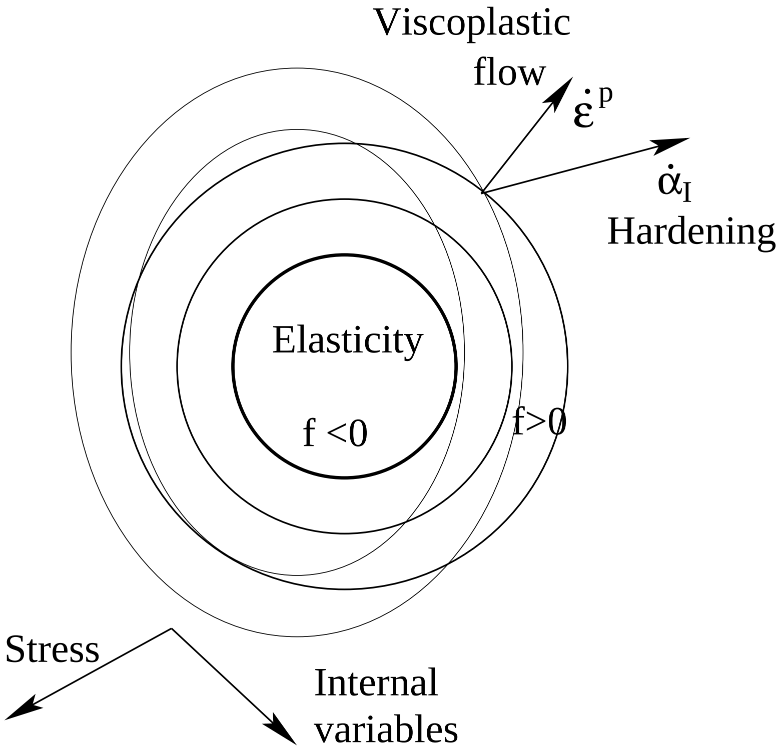
$$\frac{\partial J}{\partial \sigma_{ij}} = \frac{3}{2}\,\frac{s_{ij}}{J}$$

$$\frac{\partial J}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{3}{2}\,\frac{\underset{\sim}{\boldsymbol{s}}}{J}$$

Viscoplastic potential, associated vs standard model

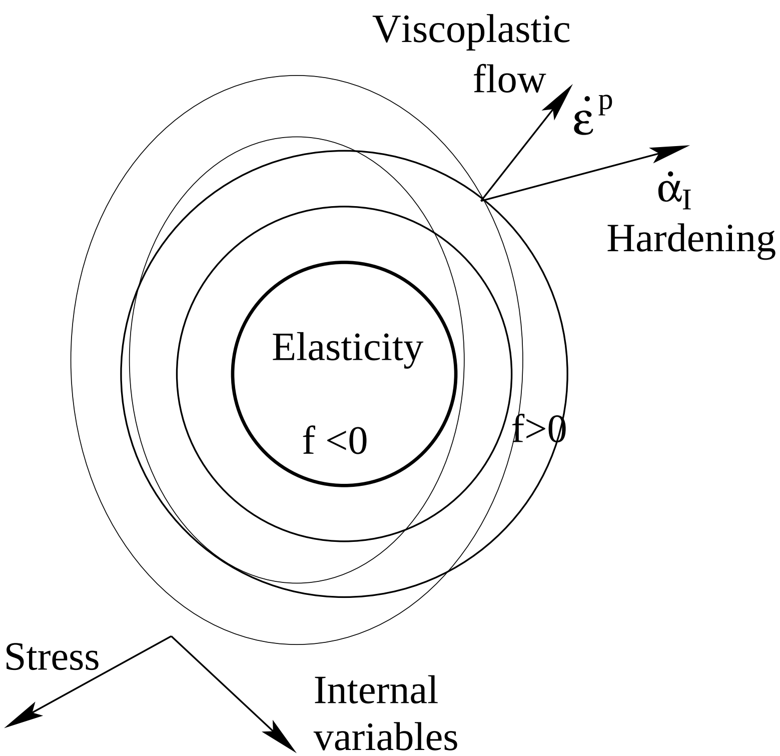


Standard model

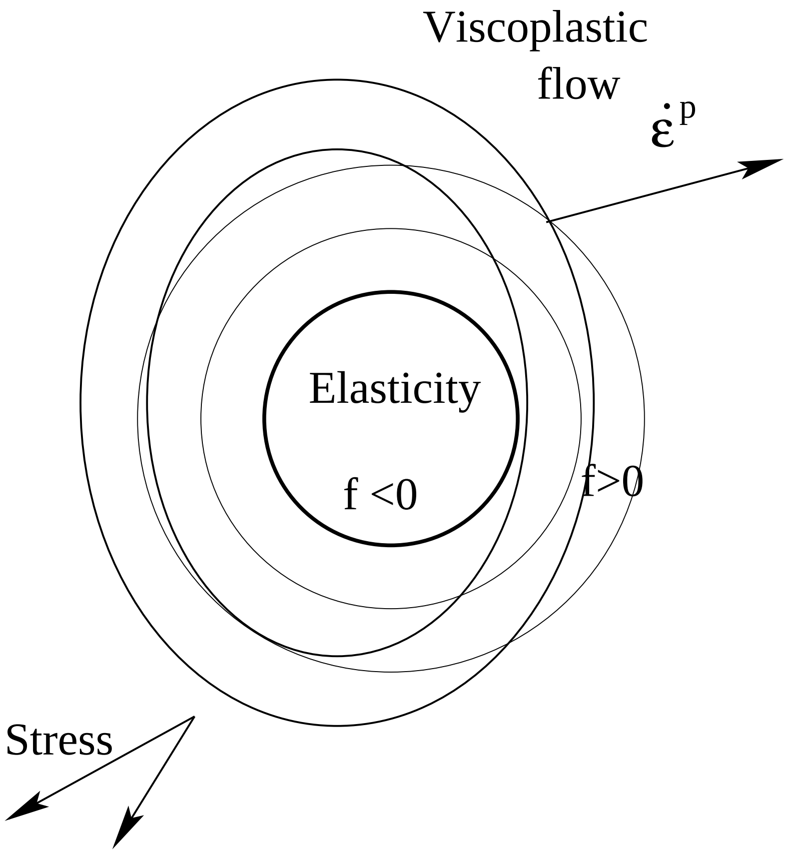


Associated model

Viscoplastic potential, general vs associated model

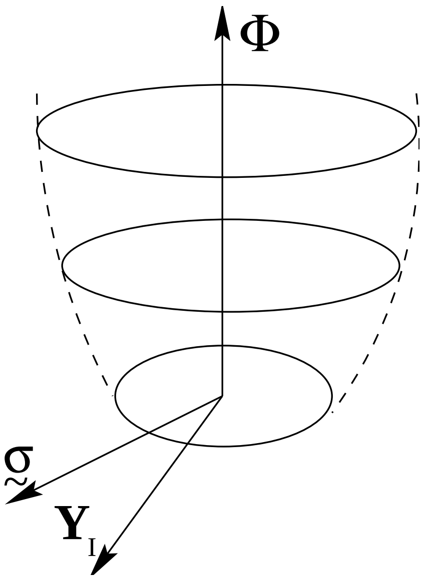


Associated model

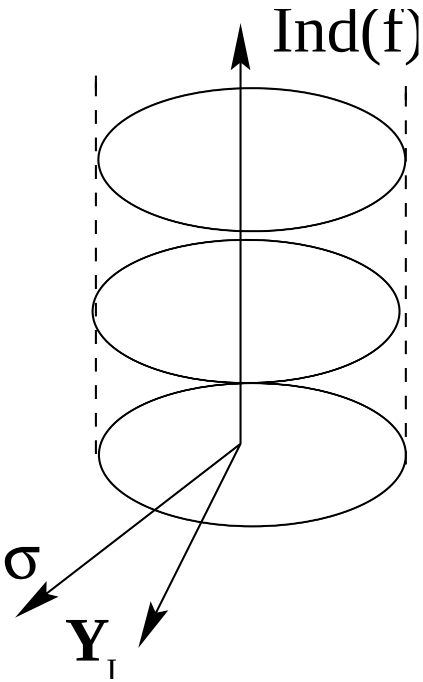


Non associated model

From viscoplasticity to plasticity



a. Viscoplastic potential



b. Plastic pseudo-potential as a limit case

- *Viscoplasticity* = after the choice of the function defining viscous effect, $\dot{v}$ is known
- *Plasticity* = $\dot{\lambda}$ to be defined from the consistency condition

Formulation of the plastic constitutive equations

$$-\text{elastic domain} \quad : \quad f(\underset{\sim}{\boldsymbol{\sigma}}, Y_I) < 0 \qquad\qquad (\dot{\underset{\sim}{\varepsilon}} = \underset{\approx}{\boldsymbol{\Lambda}}^{-1} : \dot{\underset{\sim}{\boldsymbol{\sigma}}})$$

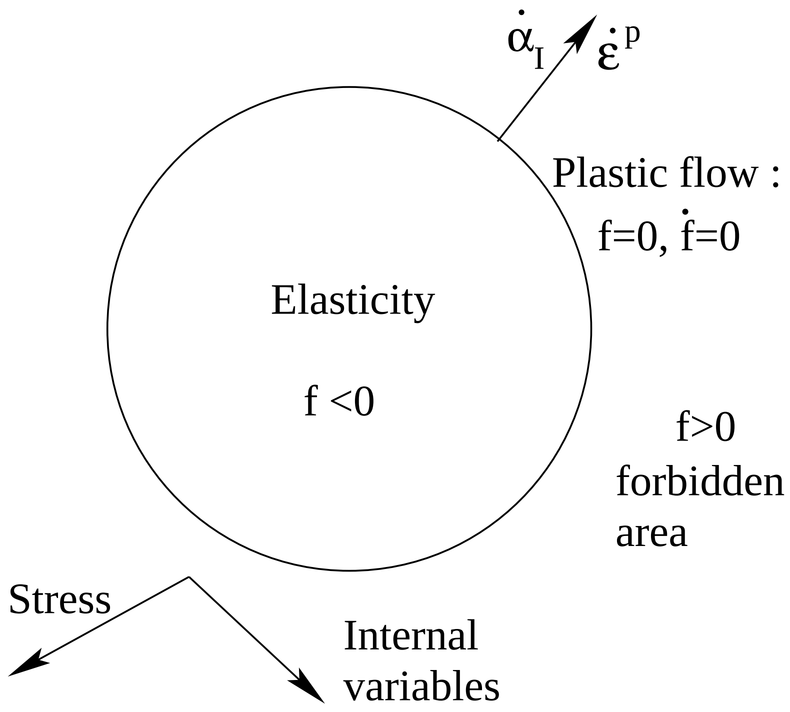
$$-\text{elastic unloading} : \quad f(\underset{\sim}{\boldsymbol{\sigma}}, Y_I) = 0 \quad \text{and } \dot{f}(\underset{\sim}{\boldsymbol{\sigma}}, Y_I) < 0 \qquad (\dot{\underset{\sim}{\varepsilon}} = \underset{\approx}{\boldsymbol{\Lambda}}^{-1} : \dot{\underset{\sim}{\boldsymbol{\sigma}}})$$

$$-\text{plastic flow} \quad : \quad f(\underset{\sim}{\boldsymbol{\sigma}}, Y_I) = 0 \quad \text{and } \dot{f}(\underset{\sim}{\boldsymbol{\sigma}}, Y_I) = 0 \qquad (\dot{\underset{\sim}{\varepsilon}} = \underset{\approx}{\boldsymbol{\Lambda}}^{-1} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} + \dot{\underset{\sim}{\varepsilon}}^p)$$

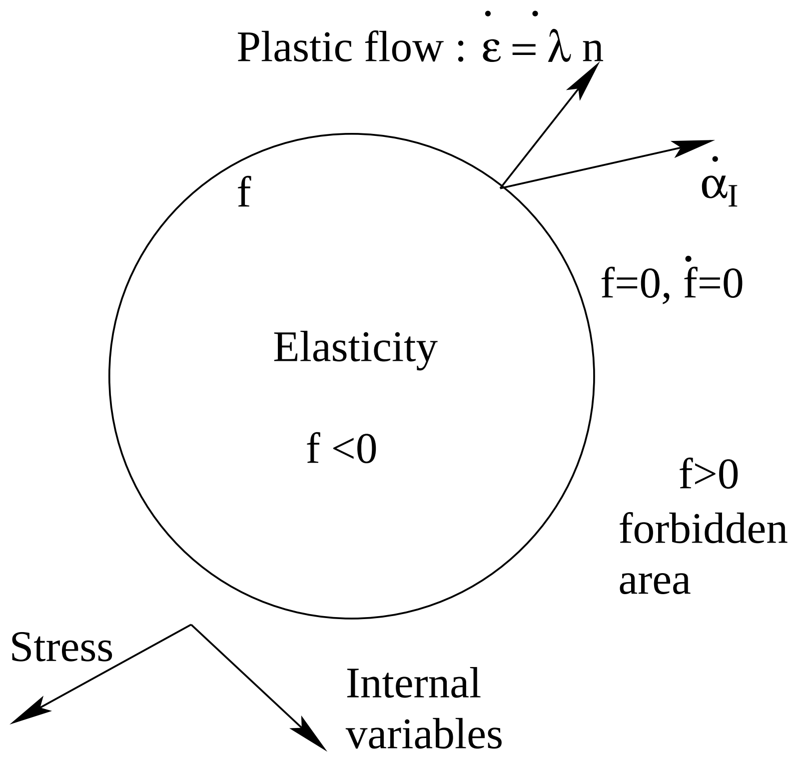
$$\dot{\underset{\sim}{\varepsilon}}^p = ...$$

$$\dot{Y}_I = ...$$

Plastic pseudo-potential, associated vs standard model

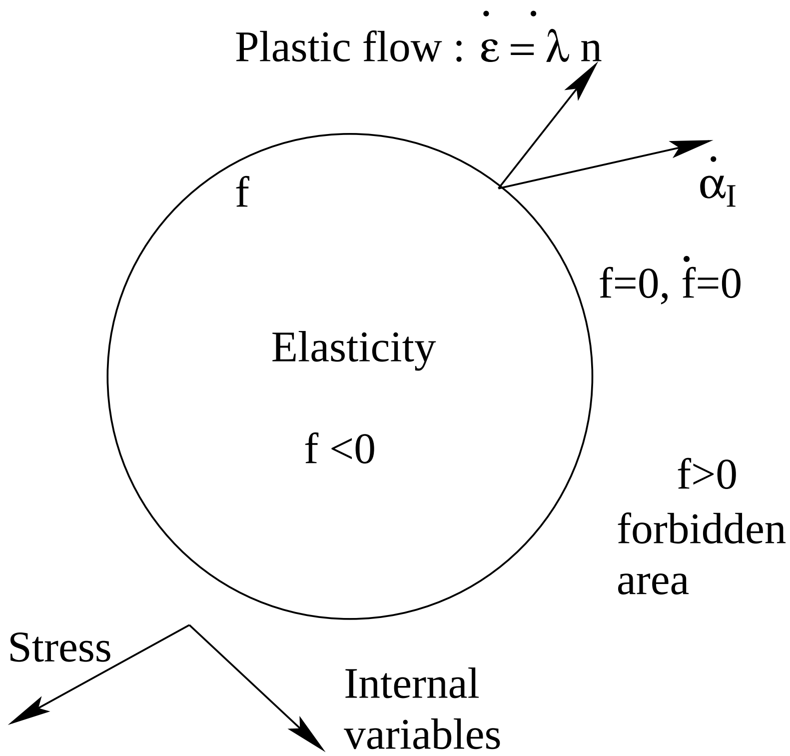


Standard model

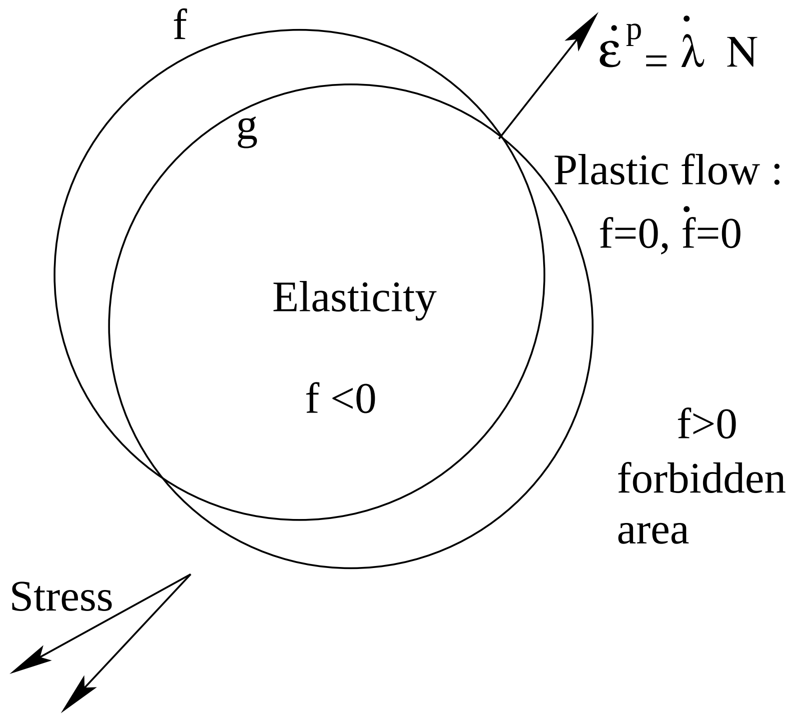


Associated model

Plastic pseudo-potential, general vs associated model



Associated model



Non associated model

Principle of maximal work

"The power of the real stress tensor $\underset{\sim}{\boldsymbol{\sigma}}$ *associated to the real plastic strain rate* $\underset{\sim}{\dot{\varepsilon}}^p$ *is larger than the power computed with any other admissible stress tensor* $\underset{\sim}{\boldsymbol{\sigma}}^*$ id est *a tensor respecting the plasticity condition associated to* $\underset{\sim}{\dot{\varepsilon}}^p$*"*. (Hill, 1951)

$$(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^*) : \underset{\sim}{\dot{\varepsilon}}^p \geq 0$$

- $\underset{\sim}{\boldsymbol{\sigma}}^*$ on the loading surface, $\underset{\sim}{\boldsymbol{\sigma}}$ in the domain, $\underset{\sim}{\dot{\varepsilon}}^p = \underset{\sim}{\mathbf{0}}$

- *Normality* rule, with $\underset{\sim}{\boldsymbol{\sigma}}^*$ in the tangent plane,

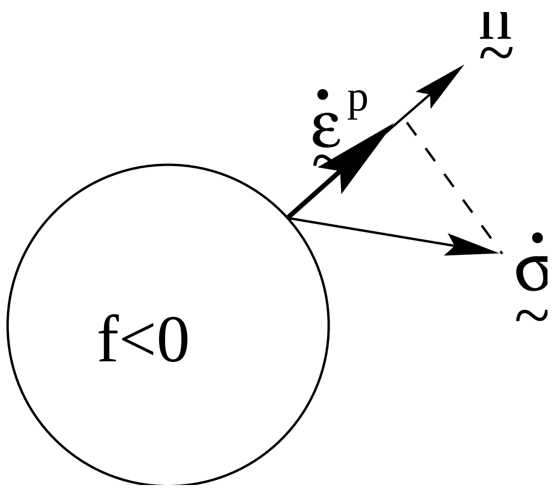
$$k\,\underset{\sim}{\boldsymbol{t}}^* : \underset{\sim}{\dot{\varepsilon}}^p \geq 0 \quad \text{and} \quad -k\,\underset{\sim}{\boldsymbol{t}}^* : \underset{\sim}{\dot{\varepsilon}}^p \geq 0$$

$$\text{so that : } \underset{\sim}{\boldsymbol{t}}^* : \underset{\sim}{\dot{\varepsilon}}^p = 0$$

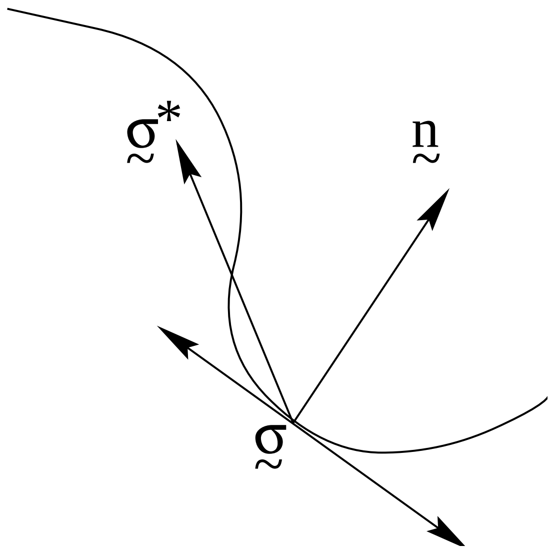
- *Sign of the multiplyier*, by setting $\underset{\sim}{\boldsymbol{\sigma}}^*$ on the interior normal, ($\underset{\sim}{\boldsymbol{\sigma}}$ on the surface),
  $(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}^*) = k\underset{\sim}{\boldsymbol{n}}$ colinear to $\underset{\sim}{\boldsymbol{n}}$ ($k > 0$), and :

$$k\,\underset{\sim}{\boldsymbol{n}} : \dot{\lambda}\underset{\sim}{\boldsymbol{n}} \geq 0 \quad \text{then :} \quad \dot{\lambda} \geq 0$$

Convexity of the loading surface



a. Illustration of the normality rule



b. Convexity of f

Perfectly plastic behavior (1)

During plastic flow, the current point representing stress state can only follow the elastic domain. The plastic multiplyier cannot be determined using stress rate

$$\text{For } f(\underset{\sim}{\boldsymbol{\sigma}}) = 0 \text{ and } \dot{f}(\underset{\sim}{\boldsymbol{\sigma}}) = 0 \quad : \quad \dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \dot{\lambda} \, \underset{\sim}{\boldsymbol{n}}$$

$$\text{During plastic flow} \quad : \quad \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} = 0$$

Perfectly plastic behavior (2)

$$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{\Lambda} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^p) \quad \text{and} \quad \underset{\sim}{n} : \dot{\underset{\sim}{\sigma}} = 0$$

$$\underset{\sim}{n} : \dot{\underset{\sim}{\sigma}} = \underset{\sim}{n} : \underset{\approx}{\Lambda} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^p) = \underset{\sim}{n} : \underset{\approx}{\Lambda} : \dot{\underset{\sim}{\varepsilon}} - \underset{\sim}{n} : \underset{\approx}{\Lambda} : \dot{\lambda}\, \underset{\sim}{n}$$

$$\dot{\lambda} = \frac{\underset{\sim}{n} : \underset{\approx}{\Lambda} : \dot{\underset{\sim}{\varepsilon}}}{\underset{\sim}{n} : \underset{\approx}{\Lambda} : \underset{\sim}{n}}$$

Case of isotropic elasticity and von Mises criterion

$$\Lambda_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad ; \quad n_{ij} = \frac{3}{2}\,\frac{s_{ij}}{J}$$

$$n_{ij}\, \Lambda_{ijkl} = 2\,\mu\, n_{kl} \quad ; \quad n_{ij}\, \Lambda_{ijkl}\, n_{kl} = 3\mu \quad ; \quad n_{ij}\, \Lambda_{ijkl}\, \dot{\varepsilon}_{kl} = 2\,\mu\, n_{kl}\, \dot{\varepsilon}_{kl}$$

$$\dot{\lambda} = \frac{2}{3}\, \underset{\sim}{n} : \dot{\underset{\sim}{\varepsilon}}$$

For onedimensional loading, with $\dot{\varepsilon} = \dot{\varepsilon}_{11}$, this last expression can be written:

$$\dot{\lambda} = \dot{\varepsilon}\, sign(\sigma) \quad \text{leading to:} \quad \dot{\varepsilon}^p = \dot{\varepsilon}$$

Flow directions associated with von Mises criterion

$f(\underset{\sim}{\boldsymbol{\sigma}}) = J(\underset{\sim}{\boldsymbol{\sigma}}) - \sigma_y$ (no hardening)

$$\underset{\sim}{\boldsymbol{n}} = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{\partial J}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \frac{\partial J}{\partial \underset{\sim}{\boldsymbol{s}}} : \frac{\partial \underset{\sim}{\boldsymbol{s}}}{\partial \underset{\sim}{\boldsymbol{\sigma}}} \quad \text{where :} \quad n_{ij} = \frac{\partial J}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}}$$

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \delta_{ik}\, \delta_{jl} - \frac{1}{3}\, \delta_{ij}\, \delta_{kl}$$

$$n_{ij} = \frac{3}{2} \frac{s_{ij}}{J} \quad \text{where:} \quad \underset{\sim}{\boldsymbol{n}} = \frac{3}{2} \frac{\underset{\sim}{\boldsymbol{s}}}{J}$$

Pure tension along direction 1 :

$$\underset{\sim}{\boldsymbol{s}} = \frac{2\,\sigma}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \quad ; \quad J = |\sigma| \quad ; \quad \underset{\sim}{\boldsymbol{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} sign(\sigma)$$

Flow directions associated with Tresca criterion

- If $\sigma_1 > \sigma_2 > \sigma_3,$: $f(\underset{\sim}{\boldsymbol{\sigma}}) = |\sigma_1 - \sigma_3| - \sigma_y,$
then $\dot{\underset{\sim}{\varepsilon}}^p_{22} = 0$ (shear type deformation) :

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- For pure tension (for instance $\sigma_1 > \sigma_2 = \sigma_3 = 0$) :

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = |\sigma_1 - \sigma_2| - \sigma_y, \text{ where } f(\underset{\sim}{\boldsymbol{\sigma}}) = |\sigma_1 - \sigma_3| - \sigma_y$$

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \dot{\mu} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Flow directions associated with Drucker–Prager criterion

$$f(\underset{\sim}{\boldsymbol{\sigma}}) = J(\underset{\sim}{\boldsymbol{\sigma}}) - \left(\sigma_y - \alpha \, \mathrm{Tr}\left(\underset{\sim}{\boldsymbol{\sigma}}\right)\right)/(1-\alpha)$$

Volume increase for any type of load:

$$\underset{\sim}{\boldsymbol{n}} = \frac{3}{2}\,\frac{\underset{\sim}{\boldsymbol{s}}}{J} + \frac{\alpha}{1-\alpha}\,\underset{\sim}{\boldsymbol{I}}$$

$$trace(\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p) = \frac{3\,\alpha}{1-\alpha}\,\dot{\lambda}$$

Prandtl-Reuss law (1)

$$f(\underset{\sim}{\boldsymbol{\sigma}}, R) = J(\underset{\sim}{\boldsymbol{\sigma}}) - \sigma_y - R(p)$$

- Hardening curve for onedimensional monotonic loading: $\sigma = \sigma_y + R(p)$.
- Plastic modulus: $H = dR/d\varepsilon^p = dR/dp$

For pure tension:

$$n_{11} = sign(\sigma) \quad , \quad n_{22} = n_{33} = (-1/2)n_{11}$$

$$\dot{\varepsilon}^p_{11} = \dot{\varepsilon}^p = sign(\sigma)\dot{\lambda} \quad , \quad \dot{\varepsilon}_{22} = \dot{\varepsilon}_{33} = (-1/2)\dot{\varepsilon}^p$$

$$\dot{p} = |\dot{\varepsilon}^p| = \dot{\lambda}$$

For general 3D case:

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \dot{\lambda}^2 \underset{\sim}{\boldsymbol{n}} : \underset{\sim}{\boldsymbol{n}} = \frac{3}{2}\dot{\lambda} \quad \text{then } \dot{p} = \left(\frac{2}{3}\,\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p\right)^{1/2}$$

Prandtl-Reuss law (2)

- Use of the consistency condition:

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} + \frac{\partial f}{\partial R}\dot{R} = 0 \quad \text{writes:} \quad \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} - H\dot{p} = 0 \quad \text{and:}$$

$$\dot{\lambda} = \frac{\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}}}{H} \text{ with } \qquad \underset{\sim}{\boldsymbol{n}} = \frac{3}{2}\frac{\underset{\sim}{\boldsymbol{s}}}{J}$$

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \dot{\lambda}\,\underset{\sim}{\boldsymbol{n}} = \frac{\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}}}{H}\,\underset{\sim}{\boldsymbol{n}}$$

For pure tension:

$$n_{11} = sign(\sigma) \quad , \quad \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} = \dot{\sigma}\, sign(\sigma) \quad \text{and:} \quad \dot{\lambda} = \dot{p} = \dot{\varepsilon}^p_{11}$$

$$\text{so that :} \quad \dot{\varepsilon}^p = \frac{n_{11}\,\dot{\sigma}}{H}\,n_{11} = \frac{\dot{\sigma}}{H}$$

Hencky-Mises Law

Assumption of *simple load*

"The applied load in terms of stresses starts from an initial virgin state, and remains proportional to one scalar parameter k"

$$\underset{\sim}{\boldsymbol{\sigma}} = k\,\underset{\sim}{\boldsymbol{\sigma}}_M \quad ; \quad \dot{\underset{\sim}{\boldsymbol{\sigma}}} = k\,\dot{\underset{\sim}{\boldsymbol{\sigma}}}_M \quad ; \quad \underset{\sim}{\boldsymbol{s}} = k\,\underset{\sim}{\boldsymbol{s}}_M \quad ; \quad J = k\,J_M \text{ with } 0 \leq k \leq 1$$

$$\underset{\sim}{\boldsymbol{n}} = \frac{3}{2} \quad \frac{\underset{\sim}{\boldsymbol{s}}_M}{J_M} \quad constant$$

$$\frac{\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}}}{H} = \frac{3}{2} \quad \frac{\underset{\sim}{\boldsymbol{\sigma}}_M}{J_M} : \underset{\sim}{\boldsymbol{\sigma}}_M \dot{k} = J_M\,\dot{k}$$

$$p = \int_0^t \dot{p}\,dt = \int_0^t \dot{\lambda} dt = \int_{k_e}^1 \frac{J_M}{H} dk$$

- axial components: $\varepsilon_{11} = (\sigma_{11} - \sigma_y)/H$
- shear components: $\frac{2}{\sqrt{3}}\,\varepsilon_{12} = (\sigma_{12}\sqrt{3} - \sigma_y)/H$

Prager rule (1)

$$f(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{X}}) = J(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{X}}) - \sigma_y \quad \text{with} \quad J(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{X}}) = \left((3/2)(\underset{\sim}{\boldsymbol{s}} - \underset{\sim}{\boldsymbol{X}}) : (\underset{\sim}{\boldsymbol{s}} - \underset{\sim}{\boldsymbol{X}})\right)^{0,5}$$

Onedimensional loading :

$$|\sigma - X| - \sigma_y = 0$$

Tensile curve modeled by:

$$\sigma = X(\varepsilon^p) + \sigma_y$$

Since $\underset{\sim}{\boldsymbol{X}}$ is proportional to $\underset{\sim}{\boldsymbol{\varepsilon}}^p$, its components for onedimensional loading are

$$X_{11}, X_{22} = X_{33} = -(1/2)X_{11}$$

Let us define:

$$\underset{\sim}{\boldsymbol{X}} = (2/3)H\underset{\sim}{\boldsymbol{\varepsilon}}^p$$

For onedimensional loading, assume:

$$X = (3/2)X_{11} = H\varepsilon^p_{11}$$

then

$$\underset{\sim}{\boldsymbol{s}} - \underset{\sim}{\boldsymbol{X}} = diag\left((2/3)\sigma - X_{11}, -(1/3)\sigma + X_{11}/2, id\right) = diag\left((2/3)(\sigma - X), -(1/3)(\sigma - X), id\right)$$

$$J(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{X}}) = |\sigma - X|$$

Prager rule (2)

Consistency condition:

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} + \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{X}}} : \dot{\underset{\sim}{\boldsymbol{X}}} = 0 \quad \text{then :} \quad \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{X}}} = 0 \quad \text{with :} \quad \underset{\sim}{\boldsymbol{n}} = \frac{3}{2} \frac{\underset{\sim}{\boldsymbol{s}} - \underset{\sim}{\boldsymbol{X}}}{J(\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{X}})}$$

$$\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{X}}} = \underset{\sim}{\boldsymbol{n}} : \frac{2}{3} H \dot{\lambda} \underset{\sim}{\boldsymbol{n}} = H \dot{\lambda} \quad \text{so that :} \quad \dot{\lambda} = (\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}})/H$$

$$\dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p = \dot{\lambda} \underset{\sim}{\boldsymbol{n}} = \frac{\underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}}}{H} \underset{\sim}{\boldsymbol{n}}$$

- Same formal expression than for isotropic hardening, nevertheless $\underset{\sim}{\boldsymbol{n}}$ is different;
- Under onedimensional loading, $\sigma = \sigma_{11}$, with $X = (3/2)X_{11}$:

$$|\sigma - X| = \sigma_y \quad , \quad \dot{\sigma} = \dot{X} = H\dot{\varepsilon}^p$$

Plastic flow under strain control

$$\dot{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\approx}{\boldsymbol{\Lambda}} : (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} - \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p) \quad \text{and :} \quad \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} = H\dot{p}$$

$$\dot{\lambda} = \frac{\underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}}{H + \underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{n}}}$$

$$\dot{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\approx}{\boldsymbol{\Lambda}} : (\dot{\underset{\sim}{\boldsymbol{\varepsilon}}} - \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}^p) = \left(\underset{\approx}{\boldsymbol{\Lambda}} - \frac{(\underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{n}}) \otimes (\underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}})}{H + \underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{n}}} \right) : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}$$

Isotropic elasticity and von Mises material:

$$\dot{\lambda} = \frac{2\mu \, \underset{\sim}{\boldsymbol{n}} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}}}{H + 3\mu}$$

Non associated plasticity

	Threshold function	Flow	Hardening
(1)	f	$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\dfrac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$	$\dot{\alpha}_I = -\dot{\lambda}\dfrac{\partial f}{\partial Y_I}$
(2)	f	$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\dfrac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$	$\dot{\alpha}_I = -\dot{\lambda}\dfrac{\partial h}{\partial Y_I}$
(3)	f	$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\dfrac{\partial g}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$	$\dot{\alpha}_I = -\dot{\lambda}\dfrac{\partial h}{\partial Y_I}$

Model (1) is standard, Model (2) is simply associated (function f is not used for determining hardening evolution, but it is still used for flow direction), the shape (3), the most general, characterizes a non-associated model

Tangent behavior in non associated plasticity

$$\underset{\sim}{\boldsymbol{n}} : \underset{\sim}{\dot{\boldsymbol{\sigma}}} + \frac{\partial f}{\partial Y_I} \dot{Y}_I = 0$$

with $\underset{\sim}{\boldsymbol{n}} = \partial f / \partial \underset{\sim}{\boldsymbol{\sigma}}$

$$\dot{\lambda} = \frac{\underset{\sim}{\boldsymbol{n}} : \underset{\sim}{\dot{\boldsymbol{\sigma}}}}{H} \text{ , with : } H = \frac{\partial f}{\partial Y_I} \frac{\partial Y_I}{\partial \alpha_I} \frac{\partial h}{\partial Y_I}$$

Assuming $\underset{\sim}{\boldsymbol{N}} = \partial g / \partial \underset{\sim}{\boldsymbol{\sigma}}$:

$$\dot{\lambda} = \frac{\underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}}{\underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{N}}}$$

$$\underset{\sim}{\dot{\boldsymbol{\sigma}}} = \left(\underset{\approx}{\boldsymbol{\Lambda}} - \frac{(\underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{N}}) \otimes (\underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}})}{H + \underset{\sim}{\boldsymbol{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\boldsymbol{N}}} \right) : \underset{\sim}{\dot{\boldsymbol{\varepsilon}}}$$

3D plasticity

- **A classical rule for (visco)plastic flow**

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\, \underset{\sim}{n}$$

- **Time independent plasticity**
  $\dot{\lambda}$ ***comes from the consistency condition***

- **Viscoplasticity**
  $\dot{\lambda}$ ***comes from the viscoplastic potential***

- $\underset{\sim}{n}$ **comes from the loading function (associated) or not (non associated) model**

–Write 3D equations for inelastic behavior–